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The space of C^1-smooth isogeometric spline functions on trilinearly parameterized volumetric two-patch domains

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Abstract

We study the C^1-smooth isogeometric spline space over trilinearly parameterized volumetric two-patch domains. Recently, the structure of this space was experimentally analyzed in [5] by numerically computing a basis and the dimension of this space. In this work, we develop the theoretical framework to explore the C^1-smooth isogeometric space. Amongst others, we use the framework to prove the numerically obtained dimension from [5] and to describe a simple explicit basis construction which consists of locally supported functions.

Keywords: Isogeometric analysis, C^1-continuity, geometric continuity, volumetric two-path domain, isogeometric basis functions

1. Introduction

Isogeometric Analysis (IGA) is an efficient instrument for a variety of engineering problems, as it facilitates discretization spaces of high order smoothness [2, 7, 13]. These discretization spaces have beneficial aspects especially when solving high order partial differential equations (PDEs). For example, C^1-smooth isogeometric spline spaces, as considered in this paper, can be used to solve fourth order PDEs such as the biharmonic equation, e.g. [1, 6, 14, 27], the Kirchhoff-Love shell problem, e.g. [3, 21, 22, 23], the Navier-Stokes-Korteweg equation [10] or the Cahn-Hilliard equation, e.g. [8, 9, 24].

The construction of C^1-smooth isogeometric spline spaces over multi-patch geometries, which are needed to describe complex physical domains, is a challenging problem, and is the task of current research, see e.g. [4, 15, 20, 28] and the mentioned references therein. The strategy followed there (and in this paper) is based on the concept of geometric continuity [12, 26], which provides a framework for the design of C^s-smooth ($s \geq 0$) isogeometric spline spaces. It uses the fact that an isogeometric function is C^s-smooth on a multi-patch domain if and only if the associated multi-patch graph surface is G^s-smooth, cf. [11].

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The goal of this paper is the theoretical investigation of the space of globally C^1-smooth isogeometric spline functions on trilinearly parameterized volumetric two-patch domains and the construction of a simple, explicitly given and locally supported basis for this space. So far, the study of C^1-smooth isogeometric spline spaces for two-patch and multi-patch domains is mostly limited to the case of bivariate functions. On the one hand, we will follow a similar approach as in [14, 19], where C^1-smooth spline spaces over bilinearly parameterized two-patch and multi-patch domains were considered and extend it to the case of trilinearly parameterized two-patch domains. On the other hand, our work will be based on the development of similar tools as introduced and used in [6, 15, 16, 17] for the investigation of bivariate C^1-smooth isogeometric spline spaces over the more general class of so-called analysis-suitable G^1 multi-patch parameterizations. This class of geometries includes the subclass of bilinearly parameterized multi-patch domains and is exactly the one which allows the design of bivariate C^1-smooth isogeometric spline spaces with optimal approximation properties [6]. Similar to [14, 15, 16, 19], we will generate basis functions which possess an explicit representation and a small local support.

The trivariate case of smooth spline spaces was considered in [5, 25, 29]. The approach in [29] explored tricubic splines on unstructured hexahedral meshes. The constructed spline spaces are C^2-smooth inside regular regions, C^1-continous across interfaces separating regular and irregular regions but only C^0-smooooth in the vicinity of extraordinary vertices. The method in [25] used a sweeping approach to generate from bivariate C^1-smooth isogeometric spaces specific trivariate ones. In [5], the entire space of C^1-smooth isogeometric spline functions on trilinearly parameterized two-patch domains was studied. More precisely, the space was numerically analyzed, which provided results on the dimension of the space obtained by interpolation. Furthermore, a basis construction for spline degree $p = 3, 4$, for regularity $r = 1$ within the single patches and for homogeneous C^1-boundary conditions was presented. The single basis functions were implicitly described by solving a (small) homogeneous system of linear equations.

In the present work, we theoretically verify the experimentally obtained dimension results from [5]. For this purpose, we develop a full theoretical framework to analyze the space of C^1-smooth isogeometric spline functions on trilinearly parameterized two-patch domains. This also allows us to generate basis functions possessing a simple explicit representation with a small local support. In addition, the construction of this basis works uniformly for arbitrary degree $p \geq 3$, regularity $1 \leq r \leq p-1$ within the single patches and is not restricted to any boundary conditions. Such a basis is an important first step towards the extension to the case of volumetric multi-patch domains by following a similar approach as in [14, 15] for bivariate spaces.

The remainder of the paper is organized as follows. Section 2 introduces the considered class of trilinearly parameterized volumetric two-patch domains and describes the space of C^1-smooth isogeometric spline functions over these domains. In Section 3, we develop the theoretical framework to investigate the C^1-smooth isogeometric spline space. This includes amongst others the decomposition of the C^1-smooth space into the direct sum of two simpler subspaces as well as the introduction of tools such as the trace and the transversal derivative of a C^1-smooth isogeometric function. In Section 4, the dimension

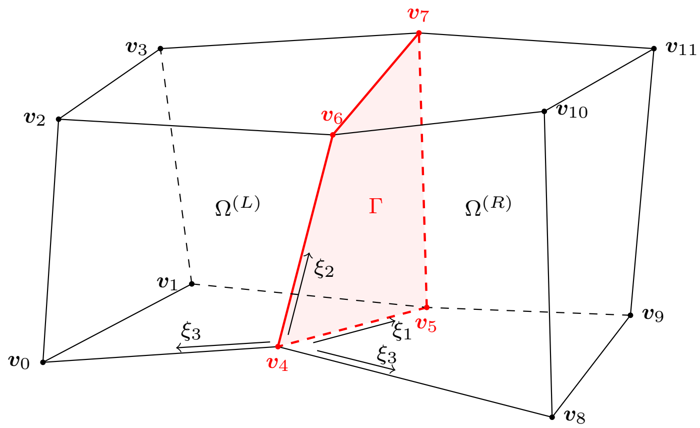

Figure 1: The two-patch domain Ω with the two trilinearly parameterized subdomains $\Omega^{(L)}, \Omega^{(R)}$, which are joined at the common interface Γ and their defining vertices.

of the space of C^1-smooth isogeometric spline functions is computed. In addition, a basis of the space is constructed, which consists of only locally supported functions possessing a simple explicit representation. Finally, we conclude the paper in Section 5.

2. The space $\mathcal{V}_{\boldsymbol{F}}$ of C^1-smooth isogeometric functions

We describe the concept of C^1-smooth isogeometric functions on trilinearly parameterized volumetric two-patch domains.

2.1. The trilinear two-patch geometry mapping $\boldsymbol{F}$

Let $\Pi^{(q_1,q_2)}$ and $\Pi^{(q_1,q_2,q_3)}$ be the space of bivariate and trivariate polynomials on $[0,1]^2$ and $[0,1]^3$ of bidegree $(q_1,q_2) \in \mathbb{Z}^2_{\geq 0}$ and tridegree $(q_1,q_2,q_3) \in \mathbb{Z}^3_{\geq 0}$, respectively. We consider a two-patch domain $\Omega = \Omega^{(L)} \cup \Omega^{(R)}$, which consists of two nondegenerate hexahedral volumetric subdomains $\Omega^{(L)}$ and $\Omega^{(R)}$ that share a common face Γ. Both subdomains are described by trilinear, regular parameterizations $\boldsymbol{F}^{(S)} : [0,1]^3 \to \Omega^{(S)}$, $S \in \{L,R\}$, which form the two-patch geometry mapping

$$\boldsymbol{F} = (\boldsymbol{F}^{(L)},\, \boldsymbol{F}^{(R)}) \in \Pi^{(1,1,1)} \times \Pi^{(1,1,1)}.$$

The two-patch domain Ω is fully determined by 12 vertices $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{11}$, which are given by

$$\boldsymbol{v}_0 = \boldsymbol{F}^{(L)}(0,0,1), \quad \boldsymbol{v}_1 = \boldsymbol{F}^{(L)}(1,0,1), \quad \boldsymbol{v}_2 = \boldsymbol{F}^{(L)}(0,1,1), \quad \boldsymbol{v}_3 = \boldsymbol{F}^{(L)}(1,1,1),$$
$$\boldsymbol{v}_4 = \boldsymbol{F}^{(S)}(0,0,0), \quad \boldsymbol{v}_5 = \boldsymbol{F}^{(S)}(1,0,0), \quad \boldsymbol{v}_6 = \boldsymbol{F}^{(S)}(0,1,0), \quad \boldsymbol{v}_7 = \boldsymbol{F}^{(S)}(1,1,0),$$

$S \in \{L,R\}$, and

$$\boldsymbol{v}_8 = \boldsymbol{F}^{(R)}(0,0,1), \quad \boldsymbol{v}_9 = \boldsymbol{F}^{(R)}(1,0,1), \quad \boldsymbol{v}_{10} = \boldsymbol{F}^{(R)}(0,1,1), \quad \boldsymbol{v}_{11} = \boldsymbol{F}^{(R)}(1,1,1),$$

see Fig. 1. This ensures that the common face Γ is parameterized by

$$\boldsymbol{F}^{(L)}(\xi_1,\xi_2,0) = \boldsymbol{F}^{(R)}(\xi_1,\xi_2,0), \quad (\xi_1,\xi_2) \in [0,1]^2.$$

2.2. The C^1-smooth isogeometric space $\mathcal{V}_{\boldsymbol{F}}$

Let $\mathcal{S}_k^{p,r}$ denote the space of spline functions on $[0,1]$ of degree $p \geq 3$ and regularity $1 \leq r \leq p-1$ with respect to the open knot vector

$$\mathcal{T}_k^{p,r} = (\underbrace{0,0,\ldots,0}_{(p+1)-\text{times}}, \underbrace{\tau_1,\tau_1,\ldots,\tau_1}_{(p-r)-\text{times}}, \ldots, \underbrace{\tau_k,\tau_k,\ldots,\tau_k}_{(p-r)-\text{times}}, \underbrace{1,1,\ldots,1}_{(p+1)-\text{times}}),$$

where $0 < \tau_i < \tau_{i+1} < 1$ for all $1 \leq i \leq k-1$ and k is the number of different inner knots. Note that the space $\mathcal{S}_k^{p,r}$ contains spline functions which are at least C^r-smooth. In addition, we denote by $N_i^{p,r}$, $i = 0,\ldots,p+k(p-r)$ the corresponding B-splines of $\mathcal{S}_k^{p,r}$ and by $\mathcal{P}$ the trivariate tensor-product spline space

$$\mathcal{P} = \mathcal{S}_k^{p,r} \otimes \mathcal{S}_k^{p,r} \otimes \mathcal{S}_k^{p,r}.$$

Furthermore, let $N_{i_1,i_2}^{p,r} = N_{i_1}^{p,r} N_{i_2}^{p,r}$, $i_1,i_2 = 0,\ldots,p+k(p-r)$ and $N_{i_1,i_2,i_3}^{p,r} = N_{i_1}^{p,r} N_{i_2}^{p,r} N_{i_3}^{p,r}$, $i_1,i_2,i_3 = 0,\ldots,p+k(p-r)$, be the tensor-product B-splines of the tensor-product spline spaces $\mathcal{S}_k^{p,r} \otimes \mathcal{S}_k^{p,r}$ and $\mathcal{S}_k^{p,r} \otimes \mathcal{S}_k^{p,r} \otimes \mathcal{S}_k^{p,r}$, respectively. Note that we also have $\boldsymbol{F}^{(S)} \in \mathcal{P}^3$ since the geometry mappings $\boldsymbol{F}^{(S)}$, $S \in \{L, R\}$, are trilinearly parameterized.

We are interested in the space $\mathcal{V}_{\boldsymbol{F}}$ of C^1-smooth isogeometric functions on Ω (with respect to the two-patch geometry $\boldsymbol{F}$ and spline space $\mathcal{P}$), i.e.

$$\mathcal{V}_{\boldsymbol{F}} = [(\mathcal{P} \times \mathcal{P}) \circ \boldsymbol{F}^{-1}] \cap C^1(\Omega).$$

Let $D_{\boldsymbol{F}} = (\beta, \gamma, \alpha^{(R)}, \alpha^{(L)})$ be the quadruple of the four bivariate polynomials $\beta, \gamma, \alpha^{(R)}$ and $\alpha^{(L)}$ given by

$$\beta(\xi_1,\xi_2) = \lambda \det\left(\partial_2 \boldsymbol{F}^{(L)}(\xi_1,\xi_2,0), \partial_3 \boldsymbol{F}^{(L)}(\xi_1,\xi_2,0), \partial_3 \boldsymbol{F}^{(R)}(\xi_1,\xi_2,0)\right),$$

$$\gamma(\xi_1,\xi_2) = \lambda \det\left(\partial_1 \boldsymbol{F}^{(L)}(\xi_1,\xi_2,0), \partial_3 \boldsymbol{F}^{(L)}(\xi_1,\xi_2,0), \partial_3 \boldsymbol{F}^{(R)}(\xi_1,\xi_2,0)\right),$$

$$\alpha^{(R)}(\xi_1,\xi_2) = \lambda \det\left(\partial_1 \boldsymbol{F}^{(R)}(\xi_1,\xi_2,0), \partial_2 \boldsymbol{F}^{(R)}(\xi_1,\xi_2,0), \partial_3 \boldsymbol{F}^{(R)}(\xi_1,\xi_2,0)\right),$$

and

$$\alpha^{(L)}(\xi_1,\xi_2) = \lambda \det\left(\partial_1 \boldsymbol{F}^{(L)}(\xi_1,\xi_2,0), \partial_2 \boldsymbol{F}^{(L)}(\xi_1,\xi_2,0), \partial_3 \boldsymbol{F}^{(L)}(\xi_1,\xi_2,0)\right),$$

respectively, where $\lambda \in \mathbb{R}_{\geq 0}$ is determined by minimizing the term

$$||\alpha^{(L)} + 1||_{L_2}^2 + ||\alpha^{(R)} - 1||_{L_2}^2.$$

Note that the two geometry mappings $\boldsymbol{F}^{(L)}$ and $\boldsymbol{F}^{(R)}$ satisfy

$$\beta\, \partial_1 \boldsymbol{F}^{(L)}|_{\xi_3=0} - \gamma\, \partial_2 \boldsymbol{F}^{(L)}|_{\xi_3=0} + \alpha^{(R)}\, \partial_3 \boldsymbol{F}^{(L)}|_{\xi_3=0} - \alpha^{(L)}\, \partial_3 \boldsymbol{F}^{(R)}|_{\xi_3=0} = 0. \tag{1}$$

Since $\boldsymbol{F}^{(L)}$ and $\boldsymbol{F}^{(R)}$ are trilinear, regular parameterizations, we obtain that

$$(\beta, \gamma, \alpha^{(R)}, \alpha^{(L)}) \in \Pi^{(3,2)} \times \Pi^{(2,3)} \times \Pi^{(2,2)} \times \Pi^{(2,2)},$$

with $\alpha^{(R)} > 0$ and $\alpha^{(L)} < 0$. The space $\mathcal{V}_{\boldsymbol{F}}$ can be characterized by

$$\mathcal{V}_{\boldsymbol{F}} = \mathcal{G}_{D_{\boldsymbol{F}}} \circ \boldsymbol{F}^{-1},$$

where $\mathcal{G}_{D_{\boldsymbol{F}}}$ is the space given by

$$\begin{aligned}\mathcal{G}_{D_{\boldsymbol{F}}} \;=\; & \Big\{ \boldsymbol{f} = (f^{(L)}, f^{(R)}) \in \mathcal{P}^2 \;:\; f^{(L)}|_{\xi_3=0} = f^{(R)}|_{\xi_3=0} \text{ and} \\ & \quad \beta\, \partial_1 f^{(L)}|_{\xi_3=0} - \gamma\, \partial_2 f^{(L)}|_{\xi_3=0} + \alpha^{(R)}\, \partial_3 f^{(L)}|_{\xi_3=0} - \alpha^{(L)}\, \partial_3 f^{(R)}|_{\xi_3=0} = 0 \Big\},\end{aligned}$$

cf. [5, 11]. This means that an isogeometric function ϕ belongs to the space $\mathcal{V}_{\boldsymbol{F}}$ if and only if the two corresponding functions $f^{(S)} = \phi \circ \boldsymbol{F}^{(S)}$, $S \in \{L, R\}$, satisfy

$$f^{(L)}|_{\xi_3=0} = f^{(R)}|_{\xi_3=0} \tag{2}$$

and

$$\beta\, \partial_1 f^{(L)}|_{\xi_3=0} - \gamma\, \partial_2 f^{(L)}|_{\xi_3=0} + \alpha^{(R)}\, \partial_3 f^{(L)}|_{\xi_3=0} - \alpha^{(L)}\, \partial_3 f^{(R)}|_{\xi_3=0} = 0. \tag{3}$$

Note that $\phi \in \mathcal{V}_{\boldsymbol{F}}$ is equivalent to the condition that the two associated graph surfaces are G^1-smooth [11].

Remark 1. In [5], $D_{\boldsymbol{F}}$ was called geometric gluing data with respect to the two-patch geometry $\boldsymbol{F}$ and $\mathcal{G}_{D_{\boldsymbol{F}}}$ was called glued spline space with respect to the geometric gluing data $D_{\boldsymbol{F}}$.

Lemma 2. *There exist bivariate functions $\beta^{(S)}$, $\gamma^{(S)}$, $S \in \{L, R\}$, such that*

$$\beta = \beta^{(R)}\, \alpha^{(L)} - \beta^{(L)}\, \alpha^{(R)} \quad \textit{and} \quad \gamma = -\gamma^{(R)}\, \alpha^{(L)} + \gamma^{(L)}\, \alpha^{(R)}. \tag{4}$$

Proof. One possible choice of the functions is

$$\beta^{(R)} = \frac{\beta}{2\alpha^{(L)}}, \quad \beta^{(L)} = -\frac{\beta}{2\alpha^{(R)}}, \quad \gamma^{(R)} = -\frac{\gamma}{2\alpha^{(L)}} \text{ and } \gamma^{(L)} = \frac{\gamma}{2\alpha^{(R)}}.$$

□

Remark 3. The functions $\beta^{(S)}$ and $\gamma^{(S)}$ in (4) are not uniquely determined. For most configurations of the trilinearly parameterized subdomains $\Omega^{(L)}$ and $\Omega^{(R)}$ the functions $\beta^{(S)}$ and $\gamma^{(S)}$ can be selected e.g. as bilinear polynomials, see Section 4.

Lemma 2 allows us to rewrite equation (3):

Lemma 4. *An isogeometric function ϕ belongs to the space $\mathcal{V}_{\boldsymbol{F}}$ if and only if the two corresponding functions $f^{(S)} = \phi \circ \boldsymbol{F}^{(S)}$, $S \in \{L, R\}$, satisfy equation* (2) *and*

$$\frac{\partial_3 f^{(L)}|_{\xi_3=0} - \beta^{(L)} \partial_1 f_0 - \gamma^{(L)} \partial_2 f_0}{\alpha^{(L)}} = \frac{\partial_3 f^{(R)}|_{\xi_3=0} - \beta^{(R)} \partial_1 f_0 - \gamma^{(R)} \partial_2 f_0}{\alpha^{(R)}} \tag{5}$$

with

$$f_0 = f^{(L)}|_{\xi_3=0} = f^{(R)}|_{\xi_3=0}. \tag{6}$$

Proof. It remains to show that equations (3) and (5) are equivalent. We can rewrite equation (3) with use of (4) and (6) as

$$\alpha^{(R)} \partial_3 f^{(L)}|_{\xi_3=0} - \beta^{(L)} \alpha^{(R)} \partial_1 f_0 - \gamma^{(L)} \alpha^{(R)} \partial_2 f_0 = \alpha^{(L)} \partial_3 f^{(R)}|_{\xi_3=0} - \beta^{(R)} \alpha^{(L)} \partial_1 f_0 - \gamma^{(R)} \alpha^{(L)} \partial_2 f_0.$$

A division by $\alpha^{(R)}\alpha^{(L)}$ on both sides of the equation yields equation (5). □

3. Decomposition of the space $\mathcal{V}_{\boldsymbol{F}}$

The space $\mathcal{V}_{\boldsymbol{F}}$ can be decomposed into the direct sum

$$\mathcal{V}_{\boldsymbol{F}} = V_{\boldsymbol{F}}^{\Omega\setminus\Gamma} \oplus V_{\boldsymbol{F}}^{\Gamma}$$

with the subspaces

$$\begin{aligned}\mathcal{V}_{\boldsymbol{F}}^{\Omega\setminus\Gamma} &= \{\phi \in \mathcal{V}_{\boldsymbol{F}} \ : \ f^{(S)}(\xi_1,\xi_2,\xi_3) = (\phi \circ \boldsymbol{F}^{(S)})(\xi_1,\xi_2,\xi_3) = \\ & \sum_{i_1=0}^{p+k(p-r)} \sum_{i_2=0}^{p+k(p-r)} \sum_{i_3=2}^{p+k(p-r)} d_{i_1,i_2,i_3}^{(S)} N_{i_1,i_2,i_3}^{p,r}(\xi_1,\xi_2,\xi_3),\ d_{i_1,i_2,i_3}^{(S)} \in \mathbb{R},\ S \in \{L,R\}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{V}_{\boldsymbol{F}}^{\Gamma} &= \{\phi \in \mathcal{V}_{\boldsymbol{F}} \ : \ f^{(S)}(\xi_1,\xi_2,\xi_3) = (\phi \circ \boldsymbol{F}^{(S)})(\xi_1,\xi_2,\xi_3) = \\ & \sum_{i_1=0}^{p+k(p-r)} \sum_{i_2=0}^{p+k(p-r)} \sum_{i_3=0}^{1} d_{i_1,i_2,i_3}^{(S)} N_{i_1,i_2,i_3}^{p,r}(\xi_1,\xi_2,\xi_3),\ d_{i_1,i_2,i_3}^{(S)} \in \mathbb{R},\ S \in \{L,R\}\}.\end{aligned} \tag{7}$$

Clearly, the space $\mathcal{V}_{\boldsymbol{F}}^{\Omega\setminus\Gamma}$ can be described as

$$\begin{aligned}\mathcal{V}_{\boldsymbol{F}}^{\Omega\setminus\Gamma} &= \{\phi = (f^{(L)}, f^{(R)}) \circ \boldsymbol{F}^{-1} \ : \ f^{(S)}(\xi_1,\xi_2,\xi_3) = \\ & \sum_{i_1=0}^{p+k(p-r)} \sum_{i_2=0}^{p+k(p-r)} \sum_{i_3=2}^{p+k(p-r)} d_{i_1,i_2,i_3}^{(S)} N_{i_1,i_2,i_3}^{p,r}(\xi_1,\xi_2,\xi_3),\ d_{i_1,i_2,i_3}^{(S)} \in \mathbb{R}, S \in \{L,R\}\},\end{aligned}$$

which implies that $\dim \mathcal{V}_{\boldsymbol{F}}^{\Omega\setminus\Gamma}$ is given by

$$\dim \mathcal{V}_{\boldsymbol{F}}^{\Omega\setminus\Gamma} = 2(p+1+k(p-r))^2(p-1+k(p-r)),$$

and that the collection of functions

$$\{\phi_{i_1,i_2,i_3}^{S}\}_{i_1,i_2=0,\ldots,p+k(p-r);i_3=2,\ldots,p+k(p-r);S\in\{L,R\}} \tag{8}$$

with

$$\phi_{i_1,i_2,i_3}^{S}(\boldsymbol{x}) = \begin{cases}(N_{i_1,i_2,i_3}^{p,r} \circ (\boldsymbol{F}^{(S)})^{-1})(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \Omega^{(S)}, \\ 0 \quad \text{otherwise,}\end{cases}$$

form a basis of $\mathcal{V}_{\boldsymbol{F}}^{\Omega\setminus\Gamma}$. To characterize the space $\mathcal{V}_{\boldsymbol{F}}^{\Gamma}$ we need the following lemma:

Lemma 5. *Let $\phi \in \mathcal{V}_{\boldsymbol{F}}^{\Gamma}$. The functions $f^{(S)} = \phi \circ \boldsymbol{F}^{(S)}$, $S \in \{L,R\}$, can be represented as*

$$\begin{aligned}f^{(S)}(\xi_1,\xi_2,\xi_3) &= f_0(\xi_1,\xi_2)\Big(N_0^{p,r}(\xi_3) + N_1^{p,r}(\xi_3)\Big) + \Big(\beta^{(S)}(\xi_1,\xi_2)\partial_1 f_0(\xi_1,\xi_2) + \\ & \gamma^{(S)}(\xi_1,\xi_2)\partial_2 f_0(\xi_1,\xi_2) + \alpha^{(S)}(\xi_1,\xi_2) f_1(\xi_1,\xi_2)\Big)\frac{\tau_1}{p} N_1^{p,r}(\xi_3),\end{aligned} \tag{9}$$

with f_0 as given in equation (6) *and*

$$f_1 = \frac{\partial_3 f^{(L)}|_{\xi_3=0} - \beta^{(L)}\partial_1 f_0 - \gamma^{(L)}\partial_2 f_0}{\alpha^{(L)}} = \frac{\partial_3 f^{(R)}|_{\xi_3=0} - \beta^{(R)}\partial_1 f_0 - \gamma^{(R)}\partial_2 f_0}{\alpha^{(R)}}. \tag{10}$$

Proof. Recall Lemma 4. We define functions $f_0, f_1 : [0,1]^2 \to \mathbb{R}$ as in equation (6) and (10), respectively. Then, the partial derivatives $\partial_3 f^{(S)}|_{\xi_3=0}$, $S \in \{L, R\}$, can be written as

$$\partial_3 f^{(S)}|_{\xi_3=0} = \beta^{(S)} \partial_1 f_0 + \gamma^{(S)} \partial_2 f_0 + \alpha^{(S)} f_1.$$

Taylor approximation of $f^{(S)}$ at $(\xi_1, \xi_2, \xi_3) = (\xi_1, \xi_2, 0)$ leads to

$$\begin{aligned} f^{(S)}(\xi_1, \xi_2, \xi_3) &= f^{(S)}(\xi_1, \xi_2, 0) + \partial_3 f^{(S)}(\xi_1, \xi_2, 0)\xi_3 + \mathcal{O}(\xi_3^2) = \\ & f_0(\xi_1, \xi_2) + \Big(\beta^{(S)}(\xi_1, \xi_2)\partial_1 f_0(\xi_1, \xi_2) + \\ & \gamma^{(S)}(\xi_1, \xi_2)\partial_2 f_0(\xi_1, \xi_2) + \alpha^{(S)}(\xi_1, \xi_2) f_1(\xi_1, \xi_2)\Big)\xi_3 + \mathcal{O}(\xi_3^2), \end{aligned}$$

which is equal to (9) by using the fact that $f^{(S)}$ is a spline function with a representation shown in (7). □

We obtain:

Theorem 6. *The space* $\mathcal{V}_{\boldsymbol{F}}^{\Gamma}$ *is equal to*

$$\begin{aligned} \mathcal{V}_{\boldsymbol{F}}^{\Gamma} &= \Big\{\phi = (f^{(L)}, f^{(R)}) \circ \boldsymbol{F}^{-1} \;:\; f^{(S)} \textit{ is given by } (9),\; f_0, f_1 : [0,1]^2 \to \mathbb{R} \\ & \textit{such that } f^{(S)} \in \mathcal{P},\; S \in \{L, R\}\Big\}. \end{aligned}$$

Proof. On the one hand if $\phi \in \mathcal{V}_{\boldsymbol{F}}^{\Gamma}$, then clearly $f^{(S)} = \phi \circ \boldsymbol{F}^{(S)} \in \mathcal{P}$, $S \in \{L, R\}$, and Lemma 5 ensures that the functions $f^{(S)}$ possess a representation (9) with functions $f_0, f_1 : [0,1]^2 \to \mathbb{R}$. On the other hand if the functions $f^{(S)}$, $S \in \{L, R\}$, possess a representation (9) then conditions (2) and (5) are fulfilled which implies by means of Lemma 4 and the fact $f^{(S)} \in \mathcal{P}$ that $\phi = (f^{(L)}, f^{(R)}) \circ \boldsymbol{F} \in \mathcal{V}_{\boldsymbol{F}}$. Since $f^{(S)}$ possess a representation of the form shown in (7) we also obtain that $\phi \in \mathcal{V}_{\boldsymbol{F}}^{\Gamma}$. □

The dimension and basis of the space $\mathcal{V}_{\boldsymbol{F}}^{\Gamma}$ will be explored in the next section. But before, let us study the functions f_0 and f_1 from the representation (9). For this purpose we will use a vector $\boldsymbol{d}$, which is transversal to Γ. This vector was already introduced in [6] for the case of planar two-patch domains and can be extended in a straightforward way to the case of volumetric two-patch domains. Let $\boldsymbol{F}_0(\xi_1, \xi_2) = \boldsymbol{F}^{(L)}(\xi_1, \xi_2, 0) = \boldsymbol{F}^{(R)}(\xi_1, \xi_2, 0)$. Then, we define the vector $\boldsymbol{d} = \boldsymbol{d}^{(L)} = \boldsymbol{d}^{(R)}$ on Γ as

$$\boldsymbol{d}^{(S)} \circ \boldsymbol{F}_0 = \Big(\partial_1 \boldsymbol{F}_0, \partial_2 \boldsymbol{F}_0, \partial_3 \boldsymbol{F}^{(S)}|_{\xi_3=0}\Big) \cdot \Big(-\beta^{(S)}, -\gamma^{(S)}, 1\Big)^T \frac{1}{\alpha^{(S)}}, \quad S \in \{L, R\}.$$

The vector $\boldsymbol{d}$ is well-defined, i.e. $\boldsymbol{d}^{(L)} = \boldsymbol{d}^{(R)}$, since the equation

$$\alpha^{(L)}\alpha^{(R)}\boldsymbol{d}^{(L)} \circ \boldsymbol{F}_0 - \alpha^{(L)}\alpha^{(R)}\boldsymbol{d}^{(R)} \circ \boldsymbol{F}_0 = 0$$

is equivalent to the system of equations (1) and (4). Furthermore, $\boldsymbol{d}$ is transversal to Γ, since

$$\det\Big(\partial_1 \boldsymbol{F}_0, \partial_2 \boldsymbol{F}_0, \boldsymbol{d}^{(S)} \circ \boldsymbol{F}_0\Big) = \frac{1}{\alpha^{(S)}} \det\Big(\partial_1 \boldsymbol{F}_0, \partial_2 \boldsymbol{F}_0, \partial_3 \boldsymbol{F}^{(S)}|_{\xi_3=0}\Big) = \frac{1}{\lambda} \neq 0.$$

Let us consider the directional derivative of $\phi \in \mathcal{V}_{\boldsymbol{F}}^{\Gamma}$ from side S with respect to $\boldsymbol{d}$ at Γ, i.e. $\nabla^{(S)}\phi \cdot \boldsymbol{d} \circ \boldsymbol{F}_0$, which is equal to

$$\left(\partial_1 f^{(S)}|_{\xi_3=0}, \partial_2 f^{(S)}|_{\xi_3=0}, \partial_3 f^{(S)}|_{\xi_3=0}\right) \cdot \left(\partial_1 \boldsymbol{F}_0, \partial_2 \boldsymbol{F}_0, \partial_3 \boldsymbol{F}^{(S)}|_{\xi_3=0}\right)^{-1} \cdot \boldsymbol{d} \circ \boldsymbol{F}_0 =$$
$$\left(\partial_1 f^{(S)}|_{\xi_3=0}, \partial_2 f^{(S)}|_{\xi_3=0}, \partial_3 f^{(S)}|_{\xi_3=0}\right) \cdot \left(-\beta^{(S)}, -\gamma^{(S)}, 1\right)^T \frac{1}{\alpha^{(S)}} =$$
$$\frac{\partial_3 f^{(S)}|_{\xi_3=0} - \beta^{(S)}\partial_1 f^{(S)}|_{\xi_3=0} - \gamma^{(S)}\partial_2 f^{(S)}|_{\xi_3=0}}{\alpha^{(S)}}.$$

Since ϕ is C^1-smooth on Ω, equation (5) is satisfied and therefore we get

$$\nabla^{(L)}\phi \cdot \boldsymbol{d} \circ \boldsymbol{F}_0 = \nabla^{(R)}\phi \cdot \boldsymbol{d} \circ \boldsymbol{F}_0 = \nabla\phi \cdot \boldsymbol{d} \circ \boldsymbol{F}_0.$$

It follows:

Corollary 7. *Let $\phi \in \mathcal{V}_{\boldsymbol{F}}^{\Gamma}$, and consider the associated representation* (9) *from the function $f^{(S)} = \phi \circ \boldsymbol{F}^{(S)}$, $S \in \{L, R\}$. The function f_0 is the trace of the function ϕ at Γ, i.e.*

$$f_0 = \phi \circ \boldsymbol{F}_0,$$

and the function f_1 is the directional derivative of the function ϕ with respect to the transversal direction $\boldsymbol{d}$ at Γ, i.e.

$$f_1 = \nabla\phi \cdot \boldsymbol{d} \circ \boldsymbol{F}_0.$$

4. Dimension and basis of the space $\mathcal{V}_{\boldsymbol{F}}^{\Gamma}$

Below, we restrict ourselves to the generic case of trilinearly parameterized two-patch domains Ω introduced in [5], which can be seen as the case valid with probability 1. For this we assume that

- the common interface Γ is nonplanar,
- the functions β, γ, $\alpha^{(R)}$ and $\alpha^{(L)}$ possess full bidegrees $(3, 2)$, $(2, 3)$, $(2, 2)$ and $(2, 2)$, respectively,
- the functions β and γ do not have roots at the values (τ_i, τ_j), $i, j = 1, \ldots, k$, where τ_i and τ_j are the inner knots of the spline space $\mathcal{S}_k^{p,r}$, and that
- the greatest common divisor of the two functions $\alpha^{(R)}$ and $\alpha^{(L)}$ is a constant function.

Note that any trilinearly parameterized two-patch domain, which does not satisfy these assumptions, can be transformed into one fulfilling the assumptions by just slightly disturbing some of the values of the vertices $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{11}$.

By investigating the functions β, γ, $\alpha^{(R)}$ and $\alpha^{(L)}$, we obtain:

Lemma 8. *There exist bilinear polynomials $\beta^{(S)}$, $\gamma^{(S)}$, $S \in \{L, R\}$ (i.e. $\beta^{(S)}, \gamma^{(S)} \in \Pi^{(1,1)}$) such that*

$$\beta = \beta^{(R)}\,\alpha^{(L)} - \beta^{(L)}\,\alpha^{(R)} \quad \text{and} \quad \gamma = -\gamma^{(R)}\,\alpha^{(L)} + \gamma^{(L)}\,\alpha^{(R)}, \tag{11}$$

with

$$\begin{aligned} \beta^{(S)}(\xi_1, \xi_2) &= \frac{\det\left(\boldsymbol{v}_7 - \boldsymbol{v}_5, \boldsymbol{v}_4 - \boldsymbol{v}_6, \partial_3 \boldsymbol{F}^{(S)}(\xi_1, \xi_2, 0)\right)}{vol} \text{ and} \\ \gamma^{(S)}(\xi_1, \xi_2) &= \frac{\det\left(\boldsymbol{v}_7 - \boldsymbol{v}_6, \boldsymbol{v}_5 - \boldsymbol{v}_4, \partial_3 \boldsymbol{F}^{(S)}(\xi_1, \xi_2, 0)\right)}{vol}, \end{aligned} \tag{12}$$

where

$$vol = \det\left(\boldsymbol{v}_5 - \boldsymbol{v}_4, \boldsymbol{v}_6 - \boldsymbol{v}_4, \boldsymbol{v}_7 - \boldsymbol{v}_4\right). \tag{13}$$

In addition, the biquadratic function $\alpha^{(S)}$, $S \in \{L, R\}$, possesses the form

$$\alpha^{(S)}(\xi_1, \xi_2) = \lambda \cdot vol \left(\delta^{(S)}(\xi_1, \xi_2) - \xi_1 \gamma^{(S)}(\xi_1, \xi_2) - \xi_2 \beta^{(S)}(\xi_1, \xi_2)\right), \tag{14}$$

where $\delta^{(S)}$, $S \in \{L, R\}$, is a bilinear function (i.e. $\delta^{(S)} \in \Pi^{(1,1)}$) given by

$$\delta^{(S)}(\xi_1, \xi_2) = \frac{\det\left(\boldsymbol{v}_5 - \boldsymbol{v}_4, \boldsymbol{v}_6 - \boldsymbol{v}_4, \partial_3 \boldsymbol{F}^{(S)}(\xi_1, \xi_2, 0)\right)}{vol}. \tag{15}$$

Proof. This can be shown by symbolic computation. □

Remark 9. The value *vol* defined in (13) is the volume of the rectangular solid spanned by the three vectors $\boldsymbol{v}_5 - \boldsymbol{v}_4, \boldsymbol{v}_6 - \boldsymbol{v}_4$ and $\boldsymbol{v}_7 - \boldsymbol{v}_4$, where $\boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_6$ and $\boldsymbol{v}_7$ are the four vertices of the common interface, see Fig. 1.

Lemma 8 allows us to study the space $\mathcal{V}_{\boldsymbol{F}}^{\Gamma}$ in more detail and to construct a basis. For this, we need some additional functions. Let $R_i^{p,r+1} : [0,1] \to \mathbb{R}$, $i = 0, \ldots, p + k(p-r-1)$, be the function defined by

$$R_i^{p,r+1}(\xi) = \frac{1}{t_{i+p} - t_i} N_{i-1}^{p-1,r}(\xi) - \frac{1}{t_{i+p+1} - t_{i+1}} N_i^{p-1,r}(\xi), \tag{16}$$

where t_i are the knots with respect to the spline space $\mathcal{S}_k^{p,r+1}$. Here and in the following of the paper, we set the term $\frac{1}{t_{i+p}-t_i}$ or $\frac{1}{t_{i+p+1}-t_{i+1}}$ to be zero, if the corresponding denominator is equal to zero. For these cases, which only happen for $i = 0$ for the first term and $i = p + k(p-r-1)$ for the second term, we also assume that the B-splines $N_{-1}^{p,r-1}$ and $N_{p+k(p-r-1)}^{p,r-1}$ are given as zero functions.

Moreover, we define the functions $\phi_{i_1,i_2}^{\Gamma,0}$, $i_1, i_2 \in \{0, \ldots, p + k(p-r-1)\}$ and $\phi_{j_1,j_2}^{\Gamma,1}$, $j_1, j_2 \in \{0, \ldots, p - 2 + k(p-r-2)\}$, as the isogeometric functions

$$((f^{(L)}, f^{(R)}) \circ \boldsymbol{F}^{-1})(\boldsymbol{x})$$

with $f^{(S)}$, $S \in \{L, R\}$, given in (9), and

$$f_0(\xi_1, \xi_2) = N^{p,r+1}_{i_1,i_2}(\xi_1, \xi_2) \text{ and } f_1(\xi_1, \xi_2) = \frac{p}{\lambda \cdot vol} R^{p,r+1}_{i_1}(\xi_1) R^{p,r+1}_{i_2}(\xi_2) \tag{17}$$

for $\phi^{\Gamma,0}_{i_1,i_2}$, and

$$f_0(\xi_1, \xi_2) = 0 \text{ and } f_1(\xi_1, \xi_2) = N^{p-2,r}_{j_1,j_2}(\xi_1, \xi_2) \tag{18}$$

for $\phi^{\Gamma,1}_{j_1,j_2}$.

Clearly, the supports of the functions $\phi^{\Gamma,0}_{i_1,i_2}$ and $\phi^{\Gamma,1}_{j_1,j_2}$ are inherited by the B-splines $N^{p-1,r}_{i}$, $N^{p,r+1}_{i_1,i_2}$ and $N^{p-2,r}_{j_1,j_2}$ used in (16), (17) and (18), respectively. Therefore, except for $r = p-1$ in case of $\phi^{\Gamma,0}_{i_1,i_2}$ and for $p-2 \leq r \leq p-1$ in case of $\phi^{\Gamma,1}_{j_1,j_2}$, the functions $\phi^{\Gamma,0}_{i_1,i_2}$ and $\phi^{\Gamma,1}_{j_1,j_2}$ possess a small local support. For the excluded cases, the B-splines $N^{p-1,r}_{i}$, $N^{p,r+1}_{i_1,i_2}$ and $N^{p-2,r}_{j_1,j_2}$ are just Bernstein polynomials defined on the interval $[0,1]$ or on the unit square $[0,1]^2$, and hence the resulting functions $\phi^{\Gamma,0}_{i_1,i_2}$ and $\phi^{\Gamma,1}_{j_1,j_2}$ possess independent of the selected number k of inner knots a support over the entire interface Γ.

Lemma 10. *The functions $\phi^{\Gamma,0}_{i_1,i_2}$, $i_1, i_2 \in \{0, \ldots, p + k(p-r-1)\}$ and $\phi^{\Gamma,1}_{j_1,j_2}$, $j_1, j_2 \in \{0, \ldots, p-2+k(p-r-2)\}$, belong to the space $\mathcal{V}^{\Gamma}_{\boldsymbol{F}}$.*

Proof. Thanks to Theorem 6, it is sufficient to prove that $\phi^{\Gamma,0}_{i_1,i_2} \circ \boldsymbol{F}^{(S)}, \phi^{\Gamma,1}_{j_1,j_2} \circ \boldsymbol{F}^{(S)} \in \mathcal{P}$, $S \in \{L, R\}$. In case of $\phi^{\Gamma,1}_{j_1,j_2}$, this can be easily seen by considering the representation (9) of the function $f^{(S)} = \phi^{\Gamma,1}_{j_1,j_2} \circ \boldsymbol{F}^{(S)}$. In case of $\phi^{\Gamma,0}_{i_1,i_2}$, it remains to show that the second term of the representation (9) of the function $f^{(S)} = \phi^{\Gamma,0}_{i_1,i_2} \circ \boldsymbol{F}^{(S)}$, i.e.

$$(\beta^{(S)}(\xi_1,\xi_2)\partial_1 f_0(\xi_1,\xi_2) + \gamma^{(S)}(\xi_1,\xi_2)\partial_2 f_0(\xi_1,\xi_2) + \alpha^{(S)}(\xi_1,\xi_2) f_1(\xi_1,\xi_2))\frac{\tau_1}{p} N^{p,r}_1(\xi_3)$$

belongs to the space $\mathcal{P}$, or simply

$$\beta^{(S)}(\xi_1,\xi_2)\partial_1 f_0(\xi_1,\xi_2) + \gamma^{(S)}(\xi_1,\xi_2)\partial_2 f_0(\xi_1,\xi_2) + \alpha^{(S)}(\xi_1,\xi_2) f_1(\xi_1,\xi_2) \in \mathcal{S}^{p,r}_k \otimes \mathcal{S}^{p,r}_k, \tag{19}$$

since the first term of (9) trivially belongs to $\mathcal{P}$.

Let us first recall the well-known recursion formula

$$N^{p,r+1}_i(\xi) = \frac{\xi - t_i}{t_{i+p} - t_i} N^{p-1,r}_{i-1}(\xi) + \frac{t_{i+p+1} - \xi}{t_{i+p+1} - t_{i+1}} N^{p-1,r}_{i}(\xi), \tag{20}$$

and the well-known derivative relation

$$\left(N^{p,r+1}_i(\xi)\right)' = p\left(\frac{1}{t_{i+p} - t_i} N^{p-1,r}_{i-1}(\xi) - \frac{1}{t_{i+p+1} - t_{i+1}} N^{p-1,r}_{i}(\xi)\right). \tag{21}$$

By means of Lemma 8 and definition (17), we get

$$\begin{gathered}\beta^{(S)}(\xi_1,\xi_2)\partial_1 f_0(\xi_1,\xi_2) + \gamma^{(S)}(\xi_1,\xi_2)\partial_2 f_0(\xi_1,\xi_2) + \alpha^{(S)}(\xi_1,\xi_2) f_1(\xi_1,\xi_2) = \\ \beta^{(S)}(\xi_1,\xi_2)\left(N^{p,r+1}_{i_1}(\xi_1)\right)' N^{p,r+1}_{i_2}(\xi_2) + \gamma^{(S)}(\xi_1,\xi_2) N^{p,r+1}_{i_1}(\xi_1)\left(N^{p,r+1}_{i_2}(\xi_2)\right)' + \\ p\,\delta^{(S)}(\xi_1,\xi_2) R^{p,r+1}_{i_1}(\xi_1) R^{p,r+1}_{i_2}(\xi_2) - \xi_1 p\,\gamma^{(S)}(\xi_1,\xi_2) R^{p,r+1}_{i_1}(\xi_1) R^{p,r+1}_{i_2}(\xi_2) - \\ \xi_2 p\,\beta^{(S)}(\xi_1,\xi_2) R^{p,r+1}_{i_1}(\xi_1) R^{p,r+1}_{i_2}(\xi_2).\end{gathered}$$

Definition (16) and relation (21) lead to

$$\beta^{(S)}(\xi_1,\xi_2)\left(N_{i_1}^{p,r+1}(\xi_1)\right)' N_{i_2}^{p,r+1}(\xi_2) + \gamma^{(S)}(\xi_1,\xi_2) N_{i_1}^{p,r+1}(\xi_1)\left(N_{i_2}^{p,r+1}(\xi_2)\right)' + \\ p\,\delta^{(S)}(\xi_1,\xi_2) R_{i_1}^{p,r+1}(\xi_1) R_{i_2}^{p,r+1}(\xi_2) - \xi_1 \gamma^{(S)}(\xi_1,\xi_2) R_{i_1}^{p,r+1}(\xi_1)\left(N_{i_2}^{p,r+1}(\xi_2)\right)' - \\ \xi_2 \beta^{(S)}(\xi_1,\xi_2)\left(N_{i_1}^{p,r+1}(\xi_1)\right)' R_{i_2}^{p,r+1}(\xi_2).$$

Using again definition (16) and the fact that the linear function ξ can be rewritten as $\xi = \xi - c + c$ for all constants $c \in \mathbb{R}$, this is equal to

$$\beta^{(S)}(\xi_1,\xi_2)\left(N_{i_1}^{p,r+1}(\xi_1)\right)' N_{i_2}^{p,r+1}(\xi_2) + \gamma^{(S)}(\xi_1,\xi_2) N_{i_1}^{p,r+1}(\xi_1)\left(N_{i_2}^{p,r+1}(\xi_2)\right)' + \\ p\,\delta^{(S)}(\xi_1,\xi_2) R_{i_1}^{p,r+1}(\xi_1) R_{i_2}^{p,r+1}(\xi_2) - \gamma^{(S)}(\xi_1,\xi_2)\Bigg(\frac{\xi_1 - t_{i_1}}{t_{i_1+p} - t_{i_1}} N_{i_1-1}^{p-1,r}(\xi_1) + \\ \frac{t_{i_1+p+1} - \xi_1}{t_{i_1+p+1} - t_{i_1+1}} N_{i_1}^{p-1,r}(\xi_1) + \frac{t_{i_1}}{t_{i_1+p} - t_{i_1}} N_{i_1-1}^{p-1,r}(\xi_1) - \frac{t_{i_1+p+1}}{t_{i_1+p+1} - t_{i_1+1}} N_{i_1}^{p-1,r}(\xi_1)\Bigg)\left(N_{i_2}^{p,r+1}(\xi_2)\right)' - \\ \beta^{(S)}(\xi_1,\xi_2)\left(N_{i_1}^{p,r+1}(\xi_1)\right)'\Bigg(\frac{\xi_2 - t_{i_2}}{t_{i_2+p} - t_j} N_{i_2-1}^{p-1,r}(\xi_2) + \frac{t_{i_2+p+1} - \xi_2}{t_{i_2+p+1} - t_{i_2+1}} N_{i_2}^{p-1,r}(\xi_2) + \\ \frac{t_{i_2}}{t_{i_2+p} - t_j} N_{i_2-1}^{p-1,r}(\xi_2) - \frac{t_{i_2+p+1}}{t_{i_2+p+1} - t_{i_2+1}} N_{i_2}^{p-1,r}(\xi_2)\Bigg).$$

With the help of the recurrence relation (20), this can be simplified to

$$p\,\delta^{(S)}(\xi_1,\xi_2) R_{i_1}^{p,r+1}(\xi_1) R_{i_2}^{p,r+1}(\xi_2) - \gamma^{(S)}(\xi_1,\xi_2)\Bigg(\frac{t_{i_1}}{t_{i_1+p} - t_{i_1}} N_{i_1-1}^{p-1,r}(\xi_1) - \\ \frac{t_{i_1+p+1}}{t_{i_1+p+1} - t_{i_1+1}} N_{i_1}^{p-1,r}(\xi_1)\Bigg)\left(N_{i_2}^{p,r+1}(\xi_2)\right)' - \beta^{(S)}(\xi_1,\xi_2)\left(N_{i_1}^{p,r+1}(\xi_1)\right)' \cdot \\ \Bigg(\frac{t_{i_2}}{t_{i_2+p} - t_{i_2}} N_{i_2-1}^{p-1,r}(\xi_2) - \frac{t_{i_2+p+1}}{t_{i_2+p+1} - t_{i_2+1}} N_{i_2}^{p-1,r}(\xi_2)\Bigg),$$

which finally implies condition (19). □

Theorem 11. *A basis of the space $\mathcal{V}_{\boldsymbol{F}}^{\Gamma}$ is given by the collection of functions*

$$\{\phi_{i_1,i_2}^{\Gamma,0}, \phi_{j_1,j_2}^{\Gamma,1}\}_{i_1,i_2 \in \{0,\ldots,p+k(p-r-1)\}; j_1,j_2 \in \{0,\ldots,p-2+k(p-r-2)\}}. \tag{22}$$

Proof. Since $\phi_{i_1,i_2}^{\Gamma,0}, \phi_{j_1,j_2}^{\Gamma,1} \in \mathcal{V}_{\boldsymbol{F}}^{\Gamma}$, compare Lemma 10 and the set of functions $\phi_{i_1,i_2}^{\Gamma,0}$ and $\phi_{j_1,j_2}^{\Gamma,1}$ are linearly independent by definition, it remains to prove that each function $\phi \in \mathcal{V}_{\boldsymbol{F}}^{\Gamma}$ can be represented as a linear combination of the functions $\phi_{i_1,i_2}^{\Gamma,0}$ and $\phi_{j_1,j_2}^{\Gamma,1}$. For this purpose, we first show that for an isogeometric function $\phi \in \mathcal{V}_{\boldsymbol{F}}^{\Gamma}$ the function f_0 from the representation (9) has to belong to the space $\mathcal{S}_k^{p,r+1} \otimes \mathcal{S}_k^{p,r+1}$, and if $f_0 = 0$, then the function f_1 has to belong to the space $\mathcal{S}_k^{p-2,r} \otimes \mathcal{S}_k^{p-2,r}$.

Let $\phi \in \mathcal{V}_{\boldsymbol{F}}^{\Gamma}$. Note that the functions $f^{(S)} = \phi \circ \boldsymbol{F}^{(S)}$, $S \in \{L, R\}$, have to fulfill equation (3), and it holds that $f_0 = f^{(L)}|_{\xi_3=0} = f^{(R)}|_{\xi_3=0}$, compare Lemma 5. Since the functions β and γ do not have roots at the values (τ_i, τ_j), $i, j = 1, \ldots, k$, see the assumptions in the beginning of this section, we obtain that $f_0 \in \mathcal{S}_k^{p,r+1} \otimes \mathcal{S}_k^{p,r+1}$. Moreover, when $f_0 = 0$ in the representation (9), then equation (10) simplifies to

$$f_1 = \frac{\partial_3 f^{(L)}|_{\xi_3=0}}{\alpha^{(L)}} = \frac{\partial_3 f^{(R)}|_{\xi_3=0}}{\alpha^{(R)}}.$$

Since $\partial_3 f^{(L)}|_{\xi_3=0}, \partial_3 f^{(R)}|_{\xi_3=0} \in \mathcal{S}_k^{p,r} \otimes \mathcal{S}_k^{p,r}$ and hence $\frac{\alpha^{(R)}\partial_3 f^{(L)}|_{\xi_3=0}}{\alpha^{(L)}}, \frac{\alpha^{(L)}\partial_3 f^{(R)}|_{\xi_3=0}}{\alpha^{(R)}} \in \mathcal{S}_k^{p,r} \otimes \mathcal{S}_k^{p,r}$, we obtain $f_1 \in \mathcal{S}_k^{p-2,r} \otimes \mathcal{S}_k^{p-2,r}$ by using the fact that $\alpha^{(L)}$ and $\alpha^{(R)}$ are biquadratic polynomials with a greatest common divisor of bidegree $(0,0)$.

Now, we are ready to show that each function $\phi \in \mathcal{V}_{\boldsymbol{F}}^{\Gamma}$ can be written as a linear combination

$$\phi(\boldsymbol{x}) = \sum_{i_1=0}^{p+k(p-r-1)} \sum_{i_2=0}^{p+k(p-r-1)} \mu_{i_1,i_2}^{\Gamma,0} \phi_{i_1,i_2}^{\Gamma,0}(\boldsymbol{x}) + \sum_{j_1=0}^{p-2+k(p-r-2)} \sum_{j_2=0}^{p-2+k(p-r-2)} \mu_{j_1,j_2}^{\Gamma,1} \phi_{j_1,j_2}^{\Gamma,1}(\boldsymbol{x}) \tag{23}$$

with factors $\mu_{i_1,i_2}^{\Gamma,0}, \mu_{j_1,j_2}^{\Gamma,1} \in \mathbb{R}$. Let $\phi \in \mathcal{V}_{\boldsymbol{F}}^{\Gamma}$ and consider the associated representation (9) from the function $f^{(S)} = \phi \circ \boldsymbol{F}^{(S)}$, $S \in \{L, R\}$. Since $f_0 \in \mathcal{S}_k^{p,r+1} \otimes \mathcal{S}_k^{p,r+1}$, the function f_0 can be written as

$$f_0(\xi_1, \xi_2) = \sum_{i_1=0}^{p+k(p-r-1)} \sum_{i_2=0}^{p+k(p-r-1)} \mu_{i_1,i_2}^{0} N_{i_1,i_2}^{p,r+1}(\xi_1, \xi_2)$$

with factors $\mu_{i_1,i_2}^{0} \in \mathbb{R}$. Computing

$$\widetilde{\phi}(\boldsymbol{x}) = \phi(\boldsymbol{x}) - \sum_{i_1=0}^{p+k(p-r-1)} \sum_{i_2=0}^{p+k(p-r-1)} \mu_{i_1,i_2}^{0} \phi_{i_1,i_2}^{\Gamma,0}(\boldsymbol{x})$$

leads to a function $\widetilde{\phi} \in \mathcal{V}_{\boldsymbol{F}}^{\Gamma}$, whose spline functions $\widetilde{f}^{(S)} = \widetilde{\phi} \circ \boldsymbol{F}^{(S)}$, $S \in \{L, R\}$, possess a representation (9) with $\widetilde{f}_0 = 0$ and $\widetilde{f}_1 \in \mathcal{S}_k^{p-2,r} \otimes \mathcal{S}_k^{p-2,r}$. Let $\mu_{j_1,j_2}^{1} \in \mathbb{R}$ be the factors such that

$$\widetilde{f}_1(\xi_1, \xi_2) = \sum_{j_1=0}^{p-2+k(p-r-2)} \sum_{j_2=0}^{p-2+k(p-r-2)} \mu_{j_1,j_2}^{1} N_{j_1,j_2}^{p-2,r}(\xi_1, \xi_2),$$

then we have that

$$\widetilde{\phi}(\boldsymbol{x}) - \sum_{j_1=0}^{p-2+k(p-r-2)} \sum_{j_2=0}^{p-2+k(p-r-2)} \mu_{j_1,j_2}^{1} \phi_{j_1,j_2}^{\Gamma,1}(\boldsymbol{x}) = 0.$$

This implies that ϕ can be written as a linear combination (23) with the factors $\mu_{i_1,i_2}^{\Gamma,0} = \mu_{i_1,i_2}^{0}$ and $\mu_{j_1,j_2}^{\Gamma,1} = \mu_{j_1,j_2}^{1}$. $\square$

We directly obtain the following two corollaries:

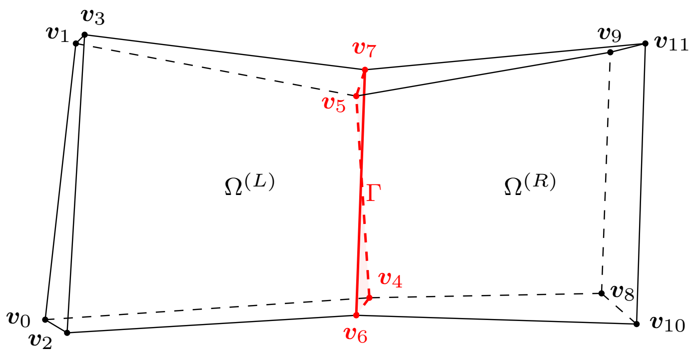

Figure 2: The generic two-patch domain Ω considered in Example 15.

Corollary 12. *The collections of functions* (8) *and* (22) *form a basis of the space* $\mathcal{V}_{\boldsymbol{F}}$*.*

Corollary 13. *It holds that*

$$\dim \mathcal{V}_{\boldsymbol{F}}^{\Gamma} = (p-1+k(p-r-2))^2 + (p+1+k(p-r-1))^2,$$

and hence

$$\begin{aligned}\dim \mathcal{V}_{\boldsymbol{F}} \quad &= \quad \underbrace{2(p+1+k(p-r))^2(p-1+k(p-r))}_{\dim \mathcal{V}_{\boldsymbol{F}}^{\Omega\setminus\Gamma}} + \\ & \qquad \underbrace{(p-1+k(p-r-2))^2 + (p+1+k(p-r-1))^2}_{\dim \mathcal{V}_{\boldsymbol{F}}^{\Gamma}}.\end{aligned}$$

Remark 14. The dimension results match with the ones in [5], where they were numerically shown for the specific choice of $r = 1$, i.e.

$$\dim \mathcal{V}_{\boldsymbol{F}}^{\Gamma} = 2 + 2k + 13k^2 \; - 10\, k(1+k)\, p \; + 2\,(k+1)^2\, p^2.$$

There, it was also demonstrated that the dimension does not change when the generic case of a planar interface is considered.

Example 15. We consider the generic trilinearly parameterized two-patch domain Ω visualized in Fig. 2, which is defined by the 12 vertices

$$\boldsymbol{v}_0 = \left(0, 0, \frac{-6}{5}\right), \quad \boldsymbol{v}_1 = \left(\frac{10}{9}, \frac{1}{10}, \frac{-32}{31}\right), \quad \boldsymbol{v}_2 = \left(\frac{1}{25}, \frac{16}{15}, \frac{-19}{20}\right), \quad \boldsymbol{v}_3 = \left(\frac{16}{15}, \frac{14}{15}, \frac{-9}{10}\right),$$

$$\boldsymbol{v}_4 = \left(\frac{1}{17}, \frac{1}{20}, \frac{2}{15}\right), \quad \boldsymbol{v}_5 = \left(\frac{13}{15}, \frac{1}{15}, \frac{2}{25}\right), \quad \boldsymbol{v}_6 = \left(\frac{1}{12}, \frac{14}{15}, \frac{1}{15}\right), \quad \boldsymbol{v}_7 = \left(\frac{18}{19}, \frac{17}{18}, \frac{2}{25}\right),$$

$$\boldsymbol{v}_8 = \left(\frac{1}{10}, \frac{1}{15}, \frac{33}{35}\right), \quad \boldsymbol{v}_9 = \left(\frac{16}{15}, \frac{1}{13}, \frac{51}{50}\right), \quad \boldsymbol{v}_{10} = \left(\frac{1}{15}, \frac{14}{15}, \frac{34}{35}\right), \quad \boldsymbol{v}_{11} = \left(\frac{26}{25}, \frac{34}{35}, \frac{21}{20}\right).$$

According to Lemma 8, we obtain

$$
\begin{aligned}
vol &= \tfrac{1086607}{21802500},\\
\beta^{(L)}(\xi_1,\xi_2) &= \tfrac{93(33064760-8289039\,\xi_2)+\xi_1\,(306334976+105798337\,\xi_2)}{2425306824},\\
\beta^{(R)}(\xi_1,\xi_2) &= \tfrac{-260(6954233+1471621\,\xi_2)+\xi_1\,(134969134+8058829\,\xi_2)}{2373149688},\\
\gamma^{(L)}(\xi_1,\xi_2) &= \tfrac{\xi_1\,(2768879480-6732851123\,\xi_2)+93(-52469680+63462141\,\xi_2)}{39168705207600},\\
\gamma^{(R)}(\xi_1,\xi_2) &= \tfrac{-260(-9716185+702889\,\xi_2)+\xi_1\,(101861470+754743721\,\xi_2)}{114979102383600},\\
\alpha^{(L)}(\xi_1,\xi_2) &= \tfrac{\xi_1^2\,(-2768879480+6732851123\,\xi_2)-10\,\xi_1\,(-9052554324+8193890948\,\xi_2+105798337\,\xi_2^2)}{486631800000}+\\
&\quad \tfrac{93(-5003665200+905234209\,\xi_2+82890390\,\xi_2^2)}{486631800000},\\
\alpha^{(R)}(\xi_1,\xi_2) &= \tfrac{-\xi_1^2\,(101861470+754743721\,\xi_2)+\xi_1\,(45101336050-23570690651\,\xi_2-80588290\,\xi_2^2)}{476166600000}+\\
&\quad \tfrac{260(1062421461+187459313\,\xi_2+14716210\,\xi_2^2)}{476166600000}.
\end{aligned}
$$

We choose for the degree $p=4$, for the regularity $r=1$ and for the number of inner knots $k=3$. Due to Corollary 13, we get

$$\dim \mathcal{V}_{\boldsymbol{F}} = \dim \mathcal{V}_{\boldsymbol{F}}^{\Omega\setminus\Gamma} + \dim \mathcal{V}_{\boldsymbol{F}}^{\Gamma} = 4704 + 157.$$

Theorem 11 provides us a basis for the resulting space $\mathcal{V}_{\boldsymbol{F}}$, which is given by the functions

$$\{\phi^{\Gamma,0}_{i_1,i_2}, \phi^{\Gamma,1}_{j_1,j_2}\}_{i_1,i_2\in\{0,\ldots,10\};j_1,j_2\in\{0,\ldots,5\}}.$$

Figure 3 illustrates two instances of basis functions, namely the functions $\phi^{\Gamma,0}_{5,5}$ and $\phi^{\Gamma,1}_{2,2}$, and show that the basis functions possess a small local support.

Remark 16. Bivariate C^1-smooth isogeometric spline spaces over the class of analysis-suitable G^1 two-patch parameterizations (which also includes the subclass of bilinearly parameterized two-patch domains) enjoy optimal approximation properties, cf. [6]. This observation was based on the fact that the traces and transversal directional derivatives along the common interface can be chosen independently from a univariate spline space of degree p and of degree $p-1$, respectively.

In contrast, in our case for the trivariate C^1-smooth space $\mathcal{V}_{\boldsymbol{F}}$, the traces (i.e. the function f_0) and the transversal directional derivatives (i.e. the function f_1) along the interface Γ can be only selected independently from a tensor-product spline space of bidegree $(p-1,p-1)$ and of bidegree $(p-2,p-2)$, respectively, which suggests at least a slight reduction of the approximation power. The restricted choice of the transversal directional derivatives f_1 was already demonstrated in the proof of Theorem 11 by showing that $f_0=0$ implies $f_1 \in \mathcal{S}_k^{p-2,r} \otimes \mathcal{S}_k^{p-2,r}$. Similarly, it could be shown for the traces f_0 that $f_1=0$ would lead to $f_0 \in \mathcal{S}_k^{p-1,r+1} \otimes \mathcal{S}_k^{p-1,r+1}$. The detailed study of the approximation power of the space $\mathcal{V}_{\boldsymbol{F}}$ is beyond the scope of the paper and deserves further investigation.

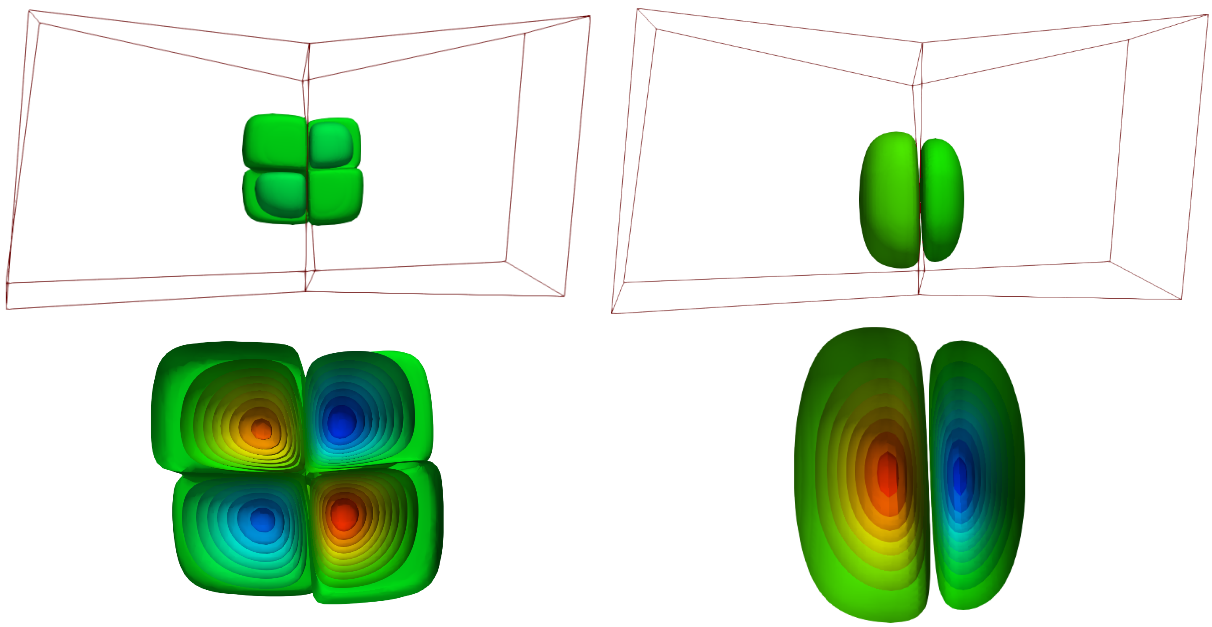

Figure 3: Examples of basis functions for $p = 4, r = 1, k = 3$. Top row: Function values visualized by isosurfaces (left: $\phi_{5,5}^{\Gamma,0}$, right: $\phi_{2,2}^{\Gamma,1}$). Bottom row: Function values visualized by level curves in a plane (left: $\phi_{5,5}^{\Gamma,0}$, right: $\phi_{2,2}^{\Gamma,1}$).

5. Conclusion

We developed the theoretical foundation to fully analyze the space $\mathcal{V}_{\boldsymbol{F}}$ of C^1-smooth isogeometric spline functions on trilinearly parameterized volumetric two-patch domains Ω. On the one hand, the dimension of the space $\mathcal{V}_{\boldsymbol{F}}$ was computed, where the obtained theoretical results verify the experimentally ones from [5]. On the other hand, a simple basis construction was presented, which works uniformly for any degree $p \geq 3$ and regularity $1 \leq r \leq p-1$. In addition, the single basis functions are locally supported (except for the case $p-2 \leq r \leq p-1$) and possess a simple explicit representation. We see this basis as an important first step for the planned extension of our approach to the case of trilinearly parameterized multi-patch domains by following e.g. a similar concept as in [14, 15] for the case of bivariate spline spaces. Moreover, the extension of our approach to more general two-patch or even multi-patch domains, comparable to the class of analysis-suitable G^1 parameterizations (cf. [6]) in 2D, is of interest, too.

In contrast to the bivariate case, e.g. [14, 18], the space $\mathcal{V}_{\boldsymbol{F}}$ of C^1-smooth isogeometric spline functions seems to have a slightly reduced approximation power. The investigation of the approximation properties of the space and the study of the magnitude of the possible reduction of the approximation power is an interesting topic for possible future work. Furthermore, we plan to explore the potential of the C^1-smooth isogeometric spline functions for applications in IGA, such as the solving of fourth order PDEs (e.g. the biharmonic equation) over volumetric two-patch and multi-patch domains.

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