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T. Takacs, B. Jüttler

G+S Report No. 4

May 2013

Der Wissenschaftsfonds.

JOHANNES KEPLER UNIVERSITY LINZ

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By Thomas Takacs, Bert Jüttler
Institute of Applied Geometry, Johannes Kepler University, Linz, Austria

Abstract

This paper is motivated by the representation of the test functions in Isogeometric Analysis (IgA). IgA is a numerical method that uses the NURBS-based representation of a CAD model to generate the finite-dimensional space of test functions which is used for the simulation. More precisely, the test functions are obtained by composing the inverse of the geometry mapping with the NURBS basis functions. We derive a representation of the derivatives of these test functions in NURBS form.

More precisely, given a (possibly piecewise) rational geometry mapping and rational test functions defined on it, we present a method to compute the derivatives of the test functions with respect to the global coordinate system. The derivatives are again given as a rational function defined on the rational geometry mapping. All computations can be described compactly using homogeneous coordinates.

We then use these results to derive conditions on the isogeometric test functions which guarantee C^k smoothness, in particular for the interesting case of singularly parametrized domains. The conditions depend heavily on the given geometry mapping. We present C^0, C^1 and C^2 smoothness results for a special class of singularly parametrized domains in $\mathbb{R}^2$ and compare them with existing H^1 and H^2 regularity results. The framework can be applied to all types of singularities and to derivatives of higher order.

1. Introduction

This work has been motivated by the isogeometric method, introduced by Hughes et al. (2005). Isogeometric analysis (IgA) directly uses the NURBS based representation of a CAD model to construct a test function space for numerical simulations. An isogeometric test function is the composition of a NURBS function with the inverse of the geometry mapping.

In many applications of IgA, the isogeometric test functions are used to perform the discretization of a partial differential equation on the physical domain. Consequently, it is necessary to evaluate the derivatives of the test functions. Fast and stable algorithms to compute the derivatives are of vital interest. As the main contribution of our paper, we provide a simple closed-form representation for the derivatives of the isogeometric test functions as parametric surface patches.

Contributions to the theoretical background of IgA include the numerical analysis concerning consistency and stability of the method Hughes et al. (2010), Echter & Bischoff (2010), Cottrell et al. (2007). The case of singularly parametrized domains is not covered by these general results; hence it has to be treated separately.

We consider singularly parametrized domains, since they are particularly useful for modeling of physical domains of general shape. Two dimensional NURBS possess a tensor-product structure, hence they can only describe quadrangular domains directly without the use of singularities. However, the smoothness of the test functions might be reduced due to the presence of a singularity.

We derive a representation of the derivatives and present conditions on the test functions that guarantee smoothness in the presence of singularities. Related results concerning the smoothness of rational singular Bézier patches can be found in Bohl & Reif (1997), Sederberg et al. (2011). Our approach provides a systematic approach to derive conditions which characterize smoothness of arbitrary order for singular surface patches.

The remainder of the paper is organized as follows. We begin with a short introduction to isogeometric analysis in Section 2.1. The remaining parts of Section 2 recall the homogeneous representation of rational Bézier patches and specify the notation used in the following. Section 3 presents an algorithmic approach to compute the derivatives of functions on rational patches with respect to the coordinates in the physical domain. In Section 4 we define a special class of singularly parametrized patches and present an algorithmic approach to analyze the C^k smoothness properties of the isogeometric test functions on this class of singular mappings. Finally we compare the C^k smoothness conditions with the H^k regularity conditions of the test functions presented in Takacs & Jüttler (2011), Takacs & Jüttler (2012).

2. Preliminaries

We start with a short introduction to the isogeometric method, which is the motivation for our work. Then we recall the notion of rational Bézier patches in homogeneous coordinates, which serves as the model case that we consider from now on.

2.1. *Motivation: Isogeometric analysis*

Isogeometric analysis (IgA) is an approach to solve partial differential equations on geometries derived from CAD systems. Since we are considering surface patches we restrict ourselves to the two-dimensional case. Nonetheless, the isogeometric method can be applied for domains in higher dimensions in a similar fashion.

The physical domain $\Omega \subset \mathbb{R}^2$ is parametrized by piecewise rational (i.e. NURBS) functions over some parameter space $\mathbf{B} \subset \mathbb{R}^2$, which is typically a box. This geometry mapping takes the form

$$\mathbf{G}(\mathbf{u}) = \sum_{\mathbf{i}} \mathbf{P}_{\mathbf{i}}\, r_{\mathbf{i},\mathbf{d}}(\mathbf{u}), \quad \mathbf{u} \in \mathbf{B}$$

with NURBS basis functions $r_{\mathbf{i},\mathbf{d}}$ of degree $\mathbf{d} = (d_1, d_2)$ and control points $\mathbf{P}_{\mathbf{i}}$. For details on rational B-splines and their applications in computer aided geometric design we refer to Hoschek & Lasser (1993), Piegl & Tiller (1995), Prautzsch, Boehm & Paluszny (2002).

A finite-dimensional space of test functions $\mathcal{V}_h$ is defined to serve as a solution space for the discretized physical problem on Ω. The basis functions spanning $\mathcal{V}_h$ are constructed from the same rational B-spline (NURBS) function space as the geometry mapping.

Let $\mathbf{G}$ satisfying $\mathbf{G}(\mathbf{B}) = \Omega$ be a mapping which is bijective in the interior of the domain. The space of isogeometric test functions on the physical domain is given by

$$\mathcal{V}_h = \operatorname*{span}_{\mathbf{i} \in \mathbb{I}} \{\varrho_{\mathbf{i},\mathbf{d}}\}, \tag{2.1}$$

where $\mathbb{I}$ is a suitable index set and the functions spanning $\mathcal{V}_h$ are defined by

$$\varrho_{\mathbf{i},\mathbf{d}} : \Omega \to \mathbb{R} : \quad (x, y) \ \mapsto \ (r_{\mathbf{i},\mathbf{d}} \circ \mathbf{G}^{-1})(x, y).$$

Here the functions $r_{\mathbf{i},\mathbf{d}}$ form a basis for the NURBS function space $\mathcal{N}_h$. In the context of

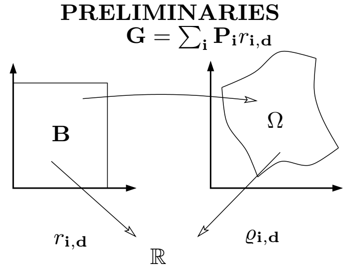

FIGURE 1. Two-dimensional geometry mapping $\mathbf{G}$ with parameter domain $\mathbf{B}$, physical domain Ω and basis functions $r_{\mathbf{i},\mathbf{d}}$ and $\varrho_{\mathbf{i},\mathbf{d}}$

isogeometric analysis the components of $\mathbf{G}$ are functions from the same NURBS space $\mathcal{N}_h$. Figure 1 summarizes these definitions of the functions $\mathbf{G}$, $r_{\mathbf{i},\mathbf{d}}$ and $\varrho_{\mathbf{i},\mathbf{d}}$.

For many applications, such as solving partial differential equations numerically or regularizing solutions of partial differential equations, it is necessary to compute derivatives of the functions $\varphi \in \mathcal{V}_h$,

$$\varphi(x, y) = \sum_{\mathbf{i}} f_{\mathbf{i}}\, \varrho_{\mathbf{i},\mathbf{d}}(x, y),$$

which are obtained as linear combinations of the functions $\varrho_{\mathbf{i},\mathbf{d}}$ with certain real coefficients $f_{\mathbf{i}}$. We present a construction of a (piecewise) rational Bézier representation for the derivatives

$$(\partial_x \varphi)(x, y) = \frac{\partial \varphi}{\partial x}(x, y) \quad \text{and} \quad (\partial_y \varphi)(x, y) = \frac{\partial \varphi}{\partial y}(x, y)$$

of these functions in Section 3. Special attention will be paid to the case of a singular mapping $\mathbf{G}$ which shall be discussed in Section 4.

2.2. *Rational functions and homogeneous coordinates*

Since we consider rational B-spline functions we can take advantage of the concept of homogeneous coordinates.

DEFINITION 1. *Every point $\mathbf{x} = [x_1, \ldots, x_n]^T$ in $\mathbb{R}^n$ can be represented in homogeneous coordinates by*

$$\tilde{\mathbf{x}} = (\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_n)^T \in \mathbb{R}^{n+1} \setminus \left\{(0, 0, \ldots, 0)^T\right\},$$

where any two points $\mathbf{x}$ and $\mathbf{y}$ are identical if and only if the homogeneous coordinate vectors $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ are linearly dependent. We shall denote this equivalence relation on the space of homogeneous coordinate vectors by

$$\tilde{\mathbf{x}} \doteq \tilde{\mathbf{y}}.$$

We extend this equivalence relation to Cartesian coordinate vectors by identifying every Cartesian coordinate vector with a special homogeneous coordinate vector via

$$[x_1, \ldots, x_n]^T = (1, x_1, \ldots, x_n)^T. \tag{2.2}$$

We shall use square brackets $[\cdots]$ and standard brackets $(\cdots)$ to identify Cartesian and homogeneous coordinate vectors, respectively.

Given a vector $\tilde{\mathbf{x}} \in \mathbb{R}^{n+1}$, we need to distinguish three cases. Either

- $\tilde{\mathbf{x}}$ is the homogeneous coordinate vector of a (finite) point, i.e. $\tilde{x}_0 \neq 0$,
- $\tilde{\mathbf{x}}$ is the homogeneous coordinate vector of a point at infinity, i.e. $\tilde{x}_0 = 0$ and $\|\tilde{\mathbf{x}}\| \neq 0$,

or

- $\tilde{\mathbf{x}}$ is a basepoint, i.e. $\tilde{\mathbf{x}} = (0, \dots, 0)^T$.

There is a well-known connection between rational parametrizations in Cartesian coordinates and polynomial parametrizations in homogeneous coordinates, since we have that

$$[x_1(\mathbf{u})/x_0(\mathbf{u}), \dots, x_n(\mathbf{u})/x_0(\mathbf{u})]^T \doteq (x_0(\mathbf{u}), x_1(\mathbf{u}), \dots, x_n(\mathbf{u}))^T,$$

where $\mathbf{u} \in \mathbf{B} \subset \mathbb{R}^2$ is the parameter and the coordinate functions x_i are polynomials.

2.3. *NURBS and rational Bézier patches in IgA*

In the following analysis we restrict ourselves to the case of rational Bézier patches in homogeneous coordinates. From this we can conclude results for general NURBS parametrizations using knot insertion and subdivision into rational Bézier segments. In the context of T-splines in IgA, the subdivision into Bézier segments is discussed in Borden et al. (2011).

Considering the case of a single bivariate Bézier patch we assume that the domain Ω is parametrized by a rational function $\mathbf{G} : \mathbf{B} \to \Omega$ with

$$\mathbf{G}(\mathbf{u}) \doteq \sum_{\mathbf{i}\in\mathbb{I}} \begin{pmatrix} w_\mathbf{i} \\ X_\mathbf{i} \\ Y_\mathbf{i} \end{pmatrix} b_{\mathbf{i},\mathbf{d}}(\mathbf{u}), \tag{2.3}$$

where

$$\mathbb{I} = \left\{ \mathbf{i} \in (\mathbb{Z}^+)^2 : \mathbf{0} \le \mathbf{i} \le \mathbf{d} \right\} \quad \text{and} \quad \mathbf{u} \in \mathbf{B} = \,]0,1[^2 .$$

The functions $b_{\mathbf{i},\mathbf{d}}(\mathbf{u})$ are tensor product Bernstein polynomials of degree $\mathbf{d}$ with

$$b_{(i_1,i_2),(d_1,d_2)}(u_1,u_2) = \begin{pmatrix} d_1 \\ i_1 \end{pmatrix} u_1^{i_1}(1-u_1)^{d_1-i_1} \begin{pmatrix} d_2 \\ i_2 \end{pmatrix} u_2^{i_2}(1-u_2)^{d_2-i_2}$$

where $\mathbf{i} = (i_1, i_2)$, $\mathbf{d} = (d_1, d_2)$ and $\mathbf{u} = (u_1, u_2)$.

3. Derivatives of functions on rational patches

We present an analytic representation of the derivatives of functions on rational Bézier patches in homogeneous coordinates.

3.1. *Representing functions in homogeneous coordinates*

In the following we consider a function φ which is defined on the domain Ω via

$$\varphi(x,y) = (\left(\frac{f}{\omega}\right) \circ \mathbf{G}^{-1})(x,y), \tag{3.1}$$

where $\mathbf{G}$ is the geometry mapping

$$\mathbf{G}(\mathbf{u}) \doteq (\omega(\mathbf{u}), X(\mathbf{u}), Y(\mathbf{u}))^T, \tag{3.2}$$

as in (2.3). The denominator ω appearing in (3.1) is the same as the denominator of the geometry mapping.

Following the isogeometric approach, the function f is represented in the same basis as the geometry mapping, i.e. f is specified as

$$f(\mathbf{u}) = \sum_{\mathbf{i}\in\mathbb{I}} f_\mathbf{i} b_{\mathbf{i},\mathbf{d}}(\mathbf{u}).$$

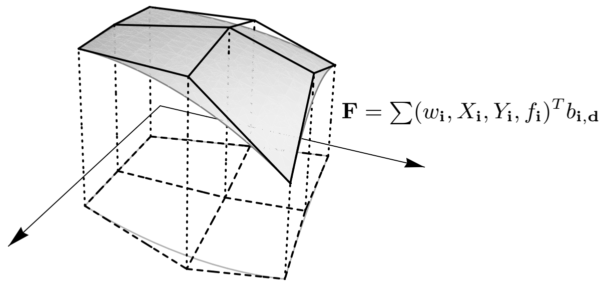

FIGURE 2. Example for a parametric rational surface patch as in (3.3) with corresponding control point grid

The function f/ω on the parameter domain $\mathbf{B}$ satisfies

$$\frac{f(\mathbf{u})}{\omega(\mathbf{u})} = \varphi\left(\frac{X(\mathbf{u})}{\omega(\mathbf{u})}, \frac{Y(\mathbf{u})}{\omega(\mathbf{u})}\right).$$

The function is represented via

$$\left[\frac{f(\mathbf{u})}{\omega(\mathbf{u})}\right] = \begin{pmatrix} 1 \\ f(\mathbf{u})/\omega(\mathbf{u}) \end{pmatrix} \doteq \sum_{\mathbf{i}\in\mathbb{I}} \begin{pmatrix} w_{\mathbf{i}} \\ f_{\mathbf{i}} \end{pmatrix} b_{\mathbf{i},\mathbf{d}}(\mathbf{u}) = \begin{pmatrix} \omega(\mathbf{u}) \\ f(\mathbf{u}) \end{pmatrix}.$$

Note that we have used the equivalence (2.2) of homogeneous and cartesian coordinates for $n = 1$ to formulate this equation.

By combining the function f and the geometry mapping $\mathbf{G}$,

$$\mathbf{F}(\mathbf{u}) = \sum_{\mathbf{i}\in\mathbb{I}} \begin{pmatrix} w_{\mathbf{i}} \\ X_{\mathbf{i}} \\ Y_{\mathbf{i}} \\ f_{\mathbf{i}} \end{pmatrix} b_{\mathbf{i},\mathbf{d}}(\mathbf{u}) \tag{3.3}$$

we represent the *graph* of the function φ (which depends on $\mathbf{x} = (x, y)$) as a *parametric rational surface patch* with parameters $\mathbf{u}$. Figure 2 presents an example of such a parametric rational surface patch. It visualizes the surface in $\mathbb{R}^3$, the underlying domain in $\mathbb{R}^2$, as well as their corresponding control point grids.

Using this representation we specified the function space $\mathcal{V}_h$ from (2.1), which consists of linear combinations of isogeometric basis functions with certain coefficients $f_{\mathbf{i}}$. In particular, we may recover the *isogeometric basis functions*, $\varphi = \varrho_{\mathbf{k},\mathbf{d}}$, by choosing Kronecker-type coefficients, $f_{\mathbf{i}} = w_{\mathbf{i}}\delta_{\mathbf{i}}^{\mathbf{k}}$ for $\mathbf{i} \in \mathbb{I}$. Here $\delta_{\mathbf{i}}^{\mathbf{k}} = \delta_{i_1}^{k_1}\delta_{i_2}^{k_2}$ is the Kronecker delta for double indices.

Note that the functions $\varphi \in \mathcal{V}_h$, the graph surface $\mathbf{F}$ as in (3.3), and the coefficient matrix $\hat{F} = (f_{\mathbf{i}})_{\mathbf{i}\in\mathbb{I}}$ all represent the same mathematical object.

3.2. *Derivatives of functions in homogeneous coordinates*

The following theorem gives a homogeneous Bézier representation of the derivatives of a function φ. To simplify the notation we write ∂_k instead of $\partial/\partial u_k$. Moreover we omit the argument of the functions and we denote with $\mathbf{V}^{[i]}$ the i-th component of the vector $\mathbf{V}$, where the indices of homogeneous coordinate vectors run from 0 to n.

THEOREM 1. *Given a function φ whose graph is represented via (3.3). Then the graphs of the partial derivatives $\partial_x\varphi$ and $\partial_y\varphi$ are described by the parametric surfaces*

$$D^x\mathbf{F} = (\omega \mathrm{J},\, X\mathrm{J},\, Y\mathrm{J},\, \omega\mathbf{H}^x \cdot (\partial_1\mathbf{H}^x \times \partial_2\mathbf{H}^x))^T \tag{3.4}$$

and

$$D^y\mathbf{F} = (\omega \mathrm{J},\, X\mathrm{J},\, Y\mathrm{J},\, \omega\mathbf{H}^y \cdot (\partial_1\mathbf{H}^y \times \partial_2\mathbf{H}^y))^T \tag{3.5}$$

respectively, where $\mathrm{J} = \mathbf{G}\cdot(\partial_1\mathbf{G}\times\partial_2\mathbf{G})$, $\mathbf{G} = (\omega, X, Y)^T$, $\mathbf{H}^x = (\omega, f, Y)^T$ *and* $\mathbf{H}^y = (\omega, X, f)^T$. *Thus, the derivatives satisfy*

$$(\partial_x\varphi)\left(\frac{X}{\omega}, \frac{Y}{\omega}\right) = \frac{(D^x\mathbf{F})^{[3]}}{(D^x\mathbf{F})^{[0]}} \quad \textit{and} \quad (\partial_y\varphi)\left(\frac{X}{\omega}, \frac{Y}{\omega}\right) = \frac{(D^y\mathbf{F})^{[3]}}{(D^y\mathbf{F})^{[0]}}.$$

We denote with D^x and D^y the operators mapping $\mathbf{F}$ onto the right hand side of (3.4) and (3.5), respectively.

Proof. Since $\varphi \circ \mathbf{G} = f$ we have

$$\nabla_{\mathbf{u}} f = (\nabla_{\mathbf{u}}\mathbf{G})^T(\nabla\varphi)(\mathbf{G}),$$

where $\nabla_{\mathbf{u}}$ denotes the gradient with respect to $\mathbf{u}$ and ∇ denotes the gradient with respect to $\mathbf{x}$. After a short calculation we arrive at

$$(\nabla\varphi)(\mathbf{G}) = (\nabla_{\mathbf{u}}\mathbf{G})^{-T}\nabla_{\mathbf{u}} f = \frac{1}{\mathrm{J}}\begin{pmatrix} \mathbf{H}^x\cdot(\partial_1\mathbf{H}^x\times\partial_2\mathbf{H}^x) \\ \mathbf{H}^y\cdot(\partial_1\mathbf{H}^y\times\partial_2\mathbf{H}^y) \end{pmatrix}.$$

This leads to

$$\left[\frac{X}{\omega}, \frac{Y}{\omega}, (\partial_x\varphi)\circ\mathbf{G}\right]^T \doteq (\omega\mathrm{J}, X\mathrm{J}, Y\mathrm{J}, \omega\mathbf{H}^x\cdot(\partial_1\mathbf{H}^x\times\partial_2\mathbf{H}^x))^T.$$

A similar representation can be derived for $\partial_y\varphi$. □

Consequently, we consider a function φ whose graph surface is described by a parametric representation $\mathbf{F}$, i.e.,

$$\mathbf{F} \doteq [x, y, \varphi]^T$$

then we can compute a similar representation for the graph surface of its partial derivatives, i.e.

$$D^x\mathbf{F} \doteq [x, y, \partial_x\varphi]^T \quad \text{and} \quad D^y\mathbf{F} \doteq [x, y, \partial_y\varphi]^T.$$

An analysis of the degrees of the rational parametric representations leads to the following result.

COROLLARY 1. *If $\mathbf{F}$ has a NURBS representation of degree $\mathbf{d} = (d_1, d_2)$, then $D^x\mathbf{F}$ and $D^y\mathbf{F}$ have a NURBS representation of degree $4\mathbf{d} - \mathbf{1} = (4d_1 - 1, 4d_2 - 1)$. However, if we consider (non-rational) B-splines of degree $\mathbf{d}$, i.e. $\omega \equiv 1$, then $D^x\mathbf{F}$ and $D^y\mathbf{F}$ have a NURBS representation of degree $3\mathbf{d} - \mathbf{1}$. In all cases, the control points of $D^x\mathbf{F}$ and $D^y\mathbf{F}$ depend linearly on the coefficients $f_{\mathbf{i}}$.*

Proof. This corollary is a direct consequence of Theorem 1. The linear dependence on the coefficients $f_{\mathbf{i}}$ follows directly from the linearity of the differentiation operator. □

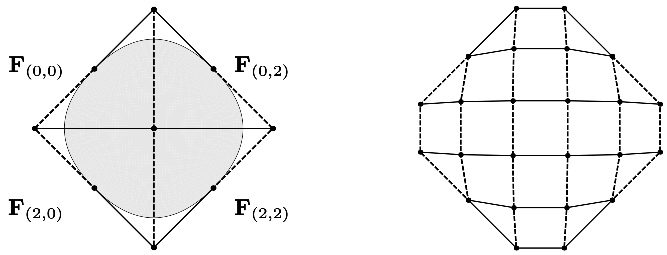

FIGURE 3. Control point grid and patch (left) for the example in Section 3.3 and of the first derivative (right).

If $\mathbf{F}$ is represented via control points

$$\mathbf{F_i} = (w_\mathbf{i}, X_\mathbf{i}, Y_\mathbf{i}, f_\mathbf{i})^T \quad \text{for} \quad \mathbf{i} \in \mathbb{I},$$

then we define the control points for $D^x\mathbf{F}$ as

$$\mathbf{F}^x_\mathbf{i} = (w^x_\mathbf{i}, X^x_\mathbf{i}, Y^x_\mathbf{i}, f^x_\mathbf{i})^T \quad \text{for} \quad \mathbf{i} \in \mathbb{I} = \left\{\mathbf{i} \in \mathbb{Z}^2 : \mathbf{0} \leq \mathbf{i} \leq 4\mathbf{d} - \mathbf{1}\right\}. \tag{3.6}$$

A similar notation is used for $D^y\mathbf{F}$ where we obtain control points $\mathbf{F}^y_\mathbf{i}$. Note that $w^x_\mathbf{i} = w^y_\mathbf{i}$, $X^x_\mathbf{i} = X^y_\mathbf{i}$, $Y^x_\mathbf{i} = Y^y_\mathbf{i}$, but $f^x_\mathbf{i} \neq f^y_\mathbf{i}$ in general.

It should be noted that the graphs of the first derivatives of a function represented by polynomial parametric surfaces are (almost) always rational surfaces. The presented computation of the derivatives can be applied iteratively to compute higher order derivatives. For instance, in order to compute the second order partial derivatives we apply the operator twice, leading to

$$D^x D^x \mathbf{F} \doteq \left[x, y, \frac{\partial^2 \varphi}{\partial x^2}\right]^T, \text{ etc.}$$

These surfaces representing the graphs of second order derivatives possess a Bézier representation of degree $4(4\mathbf{d} - 1) - 1$.

3.3. *An example*

We demonstrate the influence of the coefficients $f_\mathbf{i}$ on the control points of $D^x\mathbf{F}$ by a simple example. Consider the biquadratic patch $\mathbf{F}$ shown in Figure 3 (left) whose control points are listed in the following table:

$\mathbf{F}_{(i,j)}$	$j = 0$	$j = 1$	$j = 2$
$i = 0$	$\left(1, -1, 1, f_{(0,0)}\right)^T$	$\left(1, 0, 2, f_{(0,1)}\right)^T$	$\left(1, 1, 1, f_{(0,2)}\right)^T$
$i = 1$	$\left(1, -2, 0, f_{(1,0)}\right)^T$	$\left(1, 0, 0, f_{(1,1)}\right)^T$	$\left(1, 2, 0, f_{(1,2)}\right)^T$
$i = 2$	$\left(1, -1, -1, f_{(2,0)}\right)^T$	$\left(1, 0, -2, f_{(2,1)}\right)^T$	$\left(1, 1, -1, f_{(2,2)}\right)^T$

Due to the symmetry of the patch we consider only the derivative with respect to x. The control points of the patch $D^x\mathbf{F}$ are shown on the right-hand side in Figure 3. Since we have $\omega \equiv 1$ the degree of the derivative reduces to $(5, 5)$. The weights $w^x_\mathbf{i}$ are listed in the following table:

$w^x_{(i,j)}$	$j = 0$	$j = 1$	$j = 2$
$i = 0$	0	$\frac{24}{5}$	$\frac{36}{5}$
$i = 1$	$\frac{24}{5}$	$\frac{192}{25}$	$\frac{228}{25}$
$i = 2$	$\frac{36}{5}$	$\frac{228}{25}$	$\frac{252}{25}$

$M^x_{(i,j)}$	$j=0$	$j=1$	$j=2$
$i=0$	$\begin{pmatrix} 8 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{28}{5} & -\frac{12}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{8}{5} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{18}{5} & -\frac{4}{5} & -2 \\ \frac{2}{5} & -\frac{4}{5} & -\frac{2}{5} \\ 0 & 0 & 0 \end{pmatrix}$
$i=1$	$\begin{pmatrix} 4 & -\frac{12}{5} & 0 \\ \frac{4}{5} & -\frac{8}{5} & 0 \\ -\frac{4}{5} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{16}{5} & -\frac{44}{25} & -\frac{12}{25} \\ \frac{44}{25} & -\frac{48}{25} & \frac{44}{25} \\ -\frac{4}{25} & -\frac{8}{25} & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{56}{25} & -\frac{16}{25} & -\frac{32}{25} \\ \frac{42}{25} & -\frac{4}{5} & -\frac{26}{25} \\ \frac{2}{25} & -\frac{4}{25} & -\frac{2}{25} \end{pmatrix}$
$i=2$	$\begin{pmatrix} \frac{6}{5} & -\frac{6}{5} & 0 \\ \frac{16}{5} & -\frac{12}{5} & 0 \\ -\frac{2}{5} & -\frac{2}{5} & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{36}{25} & -\frac{6}{5} & -\frac{6}{25} \\ \frac{76}{25} & -\frac{52}{25} & -\frac{12}{25} \\ \frac{8}{25} & -\frac{18}{25} & -\frac{2}{25} \end{pmatrix}$	$\begin{pmatrix} \frac{6}{5} & -\frac{12}{25} & -\frac{18}{25} \\ \frac{58}{25} & -\frac{4}{25} & -\frac{34}{25} \\ \frac{12}{25} & -\frac{8}{25} & -\frac{8}{25} \end{pmatrix}$

TABLE 1. Some of the matrices $M^x_{(i,j)}$ for the example in Section 3.3.

The weights for $i > 2$ or $j > 2$ can be derived from the symmetry of the patch. We can write $f^x_{\mathbf{j}}$ as the Frobenius inner product, which is defined by

$$A : B = \sum_{k,\ell} A_{k,\ell} B_{k,\ell},$$

of a certain matrix $M^x_{\mathbf{j}}$ with the 3×3-matrix of coefficients $\hat{F} = (f_{\mathbf{i}})_{\mathbf{0} \le \mathbf{i} \le \mathbf{d}}$,

$$f^x_{\mathbf{j}} = M^x_{\mathbf{j}} : \hat{F},$$

for $\mathbf{j} \in \mathbb{I} = \left\{ \mathbf{j} \in \mathbb{Z}^2 : \mathbf{0} \le \mathbf{j} \le \mathbf{5} \right\}$. Table 1 lists some of the matrices $M^x_{\mathbf{j}}$.

The remaining matrices are again similar, due to the symmetry of the patch. The presented example shows that most of the matrices $M^x_{(i,j)}$ are dense, this is due to the fact that the support of each function $\varrho_{\mathbf{i},\mathbf{d}}$, corresponding to $f_{\mathbf{i}}$, covers the whole domain Ω. However, the matrices along the boundaries are not dense. Note also that the presented patch is singular in the four corner points of the parameter domain.

For a particular choice of the coefficients $f_{\mathbf{i}}$ the resulting patch for the derivative contains basepoints at the corners of the parameter domain. Such patches are used in Warren (1990) for the geometric design of more general, non-quadrangular, domains.

4. Isogeometric functions defined on singular patches

Given a rational patch with singular points, we use the previous results in order to analyze the order of smoothness of the isogeometric functions which can be defined on these patches. More precisely, we consider a rational geometry function $\mathbf{G}$ with denominator ω as in (3.2). We assume $\omega(\mathbf{u}) > 0$ for all $\mathbf{u} \in \bar{\mathbf{B}}$. Let $\det \nabla_{\mathbf{u}} \mathbf{G}(\mathbf{u}) \ge 0$ with $\det \nabla_{\mathbf{u}} \mathbf{G}(\mathbf{u}) = 0$ if and only if $\mathbf{u} \in \mathrm{D}_0 \subset \bar{\mathbf{B}}$. Consequently, D_0 denotes the set of singular points of the geometry mapping $\mathbf{G}$.

4.1. *General results*

We investigate several general properties of functions on singular patches, without taking into account the Bézier representation of the considered functions. Note that

$$\mathrm{J} = \omega^3 \det \nabla_{\mathbf{u}} \mathbf{G}$$

as defined in Theorem 1 is the Jacobian determinant of $\mathbf{G}$ multiplied by a positive weight. Since we assumed ω to be positive, the singularity of $\mathbf{G}$ can be classified based on an analysis of the zero-set D_0 of J.

LEMMA 1. *If* $\mathbf{G}$ *is a regular mapping, i.e.* $\mathrm{J}(\mathbf{u}) > 0$ *for all* $\mathbf{u} \in \bar{\mathbf{B}}$*, then* $\varphi \in C^\infty(\bar{\Omega})$ *for all* $\varphi \in \mathcal{V}_h$*.*

This is not always true for functions φ defined on singular mappings. The following proposition provides a necessary condition for C^1 smoothness.

PROPOSITION 1. *Consider a function* $\varphi \in \mathcal{V}_h$ *which corresponds to the rational parametric surface* $\mathbf{F}$*. If there exists a* $\mathbf{u} \in \bar{\mathbf{B}}$ *such that* $\mathrm{J}(\mathbf{u}) = 0$ *and*

$$(D^x\mathbf{F})^{[3]}(\mathbf{u}) \neq 0 \quad \textit{or} \quad (D^y\mathbf{F})^{[3]}(\mathbf{u}) \neq 0,$$

then $\varphi \notin C^1(\bar{\Omega})$*, where* $(D^\star\mathbf{F})^{[3]}$ *denotes the third component of* $D^\star\mathbf{F}$ *as in equation (3.4) or (3.5).*

Proof. The proof of this proposition follows directly from the properties of the function space C^1 and from Theorem 1. ☐

For non-trivial function spaces $\mathcal{V}_h$ on singularly parametrized domains it is always possible to construct such a function φ. Consequently, if singularities are present, then C^1 smoothness can be achieved only if both $D^x\mathbf{F}$ and $D^y\mathbf{F}$ possess basepoints at the singular points, i.e., $D^x\mathbf{F}(\mathbf{u}) = D^y\mathbf{F}(\mathbf{u}) = (0,0,0,0)^T$ for $\mathbf{u} \in \mathrm{D}_0$.

In the following we consider a particular class of singularly parametrized patches and analyze the derivatives of test functions defined on these singular patches. We identify the class of functions for which the Bézier representation fulfills conditions ensuring C^1 and C^2 smoothness. In the presented case the smoothness properties can be analyzed systematically. The analysis can be generalized to other classes of singular geometry mappings, but this is beyond the scope of the present paper.

4.2. *Model case: Collapsing edge*

In this section we assume that one of the edges of the control grid collapses to a single point.

DEFINITION 2. *Consider a function* ϕ *which is defined by a rational parametric surface* $\mathbf{F}$ *with the control points*

$$\mathbf{F}_{\mathbf{i}} = (w_{\mathbf{i}}, X_{\mathbf{i}}, Y_{\mathbf{i}}, f_{\mathbf{i}})^T$$

for $\mathbf{i} \in \mathbb{I}$*. We assume that the weights satisfy* $w_{\mathbf{i}} > 0$ *for all* $\mathbf{i} \in \mathbb{I}$ *and that*

$$(X_{\mathbf{i}}, Y_{\mathbf{i}})^T = \mathbf{0} \qquad \textit{if and only if } \mathbf{i} \in \mathbb{I}_0,$$

where $\mathbb{I}_0 = \{(i,j) \in \mathbb{I} : i = 0\}$*. Furthermore, we assume that the mapping*

$$\mathbf{u} \mapsto [X(\mathbf{u})/\omega(\mathbf{u}), Y(\mathbf{u})/\omega(\mathbf{u})]^T$$

is singular if and only if $\mathbf{u} \in D_0 = \{0\} \times [0,1]$*. We say that such a function* φ *is a function on an* ***edge-degenerated*** *patch.*

Considering a patch with a degenerated edge we recall the well-known condition on the control points that guarantees the continuity of the function.

PROPOSITION 2. *Let* $\mathbf{F}$ *be a parametric surface representing the graph of a function* φ *on an edge-degenerated patch. Then the function* $\varphi \in \mathcal{V}_h$ *corresponding to* $\mathbf{F}$ *is continuous,* *i.e.* $\varphi \in C^0(\Omega)$, *if and only if the control points* $\mathbf{F_i}$ *are linearly dependent for all* $\mathbf{i} \in \mathbb{I}_0$. *This is equivalent to*

$$f_{\mathbf{i}} = \lambda w_{\mathbf{i}} \ \ \forall \, \mathbf{i} \in \mathbb{I}_0$$

for some $\lambda \in \mathbb{R}$.

This proposition is one ingredient for the higher order smoothness analysis of isogeometric test functions. Another ingredient is described on the proposition below which describes possible cancellation of basepoints. It can be used under the conditions formulated in the following assumption.

Definition 3. *Consider a parametric surface* $\mathbf{F}$ *with control points* $\mathbf{F_i}$ *for* $\mathbf{i} \in \mathbb{I}$. *It is said to satisfy the* ***cancellation assumption*** *if*

- *all points* $\mathbf{F_i}$ *correspond to real points for* $\mathbf{i} \in \mathbb{I}\backslash\mathbb{I}_0$ *and*
- *all points* $\mathbf{F_i}$ *be basepoints for* $\mathbf{i} \in \mathbb{I}_0$,

where $\mathbb{I}_0 = \{(i,j) \in \mathbb{I} : \, i = 0\}$.

In this case the following fact is also well-known.

Lemma 2. *Consider a function* $\mathbf{F}$ *of degree* (d_1, d_2) *which fulfills the cancellation assumption. Then there exists a polynomial parametrization in homogeneous coordinates* $\mathbf{F}^* = (1/u_1)\mathbf{F}$ *of degree* $(d_1 - 1, d_2)$ *with*

$$\mathbf{F}(\mathbf{u}) \doteq \mathbf{F}^*(\mathbf{u}) = \frac{1}{u_1}\mathbf{F}(\mathbf{u}).$$

where $\mathbf{u} = (u_1, u_2)$. *The control points of* $\mathbf{F}^*$ *depend linearly on the non-zero control points of* $\mathbf{F}$. *More precisely, we have that* $\mathbf{F}^*_{\mathbf{i}-(1,0)} = (d_1/i_1)\mathbf{F_i}$ *for* $\mathbf{i} \in \mathbb{I}\backslash\mathbb{I}_0$.

We can arrange the homogeneous control points $\mathbf{F_i}$ in a matrix of size $(d_1{+}1)\times(d_2{+}1)$. The proposition states that if the first row of the matrix contains only basepoints, then this row can be cancelled out entirely.

In order to analyze C^1 smoothness of functions defined on singular mappings we need some preparations. Let

$$\hat{F} = \begin{pmatrix} f_{(0,0)} & \cdots & f_{(0,d_2)} \\ \vdots & \ddots & \vdots \\ f_{(d_1,0)} & \cdots & f_{(d_1,d_2)} \end{pmatrix} \tag{4.1}$$

be the matrix of all coefficients $f_{\mathbf{i}} \in \mathbb{I}$. We know that the control points of $D^x\mathbf{F}$ and $D^y\mathbf{F}$ depend linearly on the coefficients $f_{\mathbf{i}}$, hence we can find matrices $M^x_{\mathbf{i}}$ and $M^y_{\mathbf{i}}$ such that

$$M^x_{\mathbf{j}} : \hat{F} = f^x_{\mathbf{j}} \ \text{ and } \ M^y_{\mathbf{j}} : \hat{F} = f^y_{\mathbf{j}} \tag{4.2}$$

for $\mathbf{j} \in \mathbb{I} = \{(i,j) \in \mathbb{Z}^2 : \mathbf{0} \le (i,j) \le 4\mathbf{d} - \mathbf{1}\}$. Now we can prove the following result.

Theorem 2. *Consider a function* $\varphi \in \mathcal{V}_h$ *on an edge-degenerated patch and let* $\mathbf{F}$ *be the parametrization of the graph surface, There exists a* $\lambda^x \in \mathbb{R}$, *such that* $\hat{F}$ *solves*

$$\begin{aligned} M^x_{\mathbf{i}} : \hat{F} \ &= \ \ 0 \qquad \textit{for} \ \ \mathbf{i} \in \mathbb{I}_0 = \{(0,j) \in \mathbb{I}\} \ \ \textit{and} \\ M^x_{\mathbf{j}} : \hat{F} \ &= \lambda^x w^x_{\mathbf{j}} \quad \textit{for} \ \ \mathbf{j} \in \mathbb{I}_1\{(1,j) \in \mathbb{I}\} \end{aligned} \tag{4.3}$$

if and only if $\partial_x\varphi \in C^0(\bar{\Omega})$. *Hence* $\varphi \in C^1(\bar{\Omega})$ *if and only if* $\hat{F}$ *fulfills (4.3) for* x *and an equivalent system for* y. *Note that the coefficients of the linear system (4.3) for* $\hat{F}$ *and* λ^x *are completely defined via equations* (3.4), (3.5), (3.6) *and* (4.2) *and the cancellation described in Lemma 2.*

Proof. The representation derived in Theorem 1 leads to

$$D^x\mathbf{F} \doteq \left(\omega \mathrm{J}, X\mathrm{J}, Y\mathrm{J}, \omega \mathbf{H}^x \cdot (\partial_1 \mathbf{H}^x \times \partial_2 \mathbf{H}^x)\right)^T.$$

The control points fulfill $\mathbf{F}^x_{\mathbf{i}} = (0,0,0,f^x_{\mathbf{i}})^T$ for all $\mathbf{i} \in \mathbb{I}_0$. Hence $D^x\mathbf{F}$ fulfills the cancellation assumption (Definition 3) if and only if $f^x_{\mathbf{i}} = 0$ for all $\mathbf{i} \in \mathbb{I}_0$, which is equivalent to

$$M^x_{\mathbf{i}} : \hat{F} = 0 \quad \text{for} \quad \mathbf{i} \in \mathbb{I}_0.$$

We now apply Lemma 2 and obtain a reduced function $D^x\mathbf{F}/u_1$ which is again a function on an edge-degenerated patch (Definition 2). Now we can apply Proposition 2 to the reduced function, which gives the condition

$$f^x_{\mathbf{j}} = \lambda^x w^x_{\mathbf{j}},$$

for a certain constant λ^x and for all $\mathbf{j} \in \mathbb{I}_1$. This is equivalent to

$$M^x_{\mathbf{j}} : \hat{F} = \lambda^x w^x_{\mathbf{j}},$$

which concludes the proof. □

This theorem can be applied to analyze higher order derivatives as well. To prove C^2 smoothness one has to start with a homogeneous Bézier representation of $\partial_x\varphi$ and $\partial_y\varphi$ and derive conditions for the second derivatives $\partial^2_x\varphi$, $\partial_x\partial_y\varphi$ and $\partial^2_y\varphi$.

The presented results for C^1 smoothness are equivalent to the well-known condition that $\varphi \in C^1(\bar{\Omega})$ if and only if all

$$\mathbf{F}_{\mathbf{i}}, \quad \text{for} \ \ \mathbf{i} \in \{(i,j) \in \mathbb{I} : \, 0 \le i \le 1\},$$

lie in a common plane.

In the following we present examples of edge-degenerated patches as in Assumption 2 and derive C^0, C^1 and C^2 smoothness conditions for general functions $\varphi \in \mathcal{V}_h$.

4.3. *Study of continuity on example patches*

We present two example patches and compare the results. We start with a general biquadratic patch in Example 1 and present a degree-elevated bilinear patch in Example 2.

EXAMPLE 1. *Consider a biquadratic patch ($\mathbf{d} = (2,2)$) with the control points listed in the following table, where the coefficients $f_{(i,j)} \in \mathbb{R}$ can be chosen arbitrarily.*

$\mathbf{F}_{(i,j)}$	$j=0$	$j=1$	$j=2$
$i=0$	$\left(1,0,0,f_{(0,0)}\right)^T$	$\left(1,0,0,f_{(0,1)}\right)^T$	$\left(1,0,0,f_{(0,2)}\right)^T$
$i=1$	$\left(1,\frac{1}{2},0,f_{(1,0)}\right)^T$	$\left(1,\frac{1}{2},\frac{1}{2},f_{(1,1)}\right)^T$	$\left(1,0,\frac{1}{2},f_{(1,2)}\right)^T$
$i=2$	$\left(1,1,0,f_{(2,0)}\right)^T$	$\left(1,1,1,f_{(2,1)}\right)^T$	$\left(1,0,1,f_{(2,2)}\right)^T$

Figure 4 (left) shows the control point grid of this patch.

Let $\hat{F}$ be the matrix of coefficients as in (4.1). It is clear that every matrix $\hat{F} \in \mathbb{R}^{3\times 3}$ corresponds to a function $\varphi \in \mathcal{V}_h$ and vice versa.

In this case we have

$$\mathrm{J}(u_1,u_2) = u_1(2 - 2u_2 + 2{u_2}^2)$$

with zero set $D_0 = \{\mathbf{u} : u_1 = 0\}$. Let $\star = x$ or $\star = y$. The control points of the derivatives

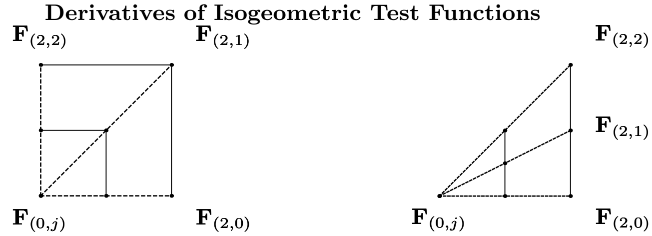

FIGURE 4. Control point grid for Example 1 (left) and 2 (right)

are listed in the following table:

$D^\star \mathbf{F}_{(i,j)}$	$j=0$	$j=1$	$\dots$	$j=7$
$i=0$	$(0,0,0,*)^T$	$(0,0,0,*)^T$	$\dots$	$(0,0,0,*)^T$
$i=1$	$\left(\frac{2}{7},0,0,*\right)^T$	$\left(\frac{12}{49},0,0,*\right)^T$	$\dots$	$\left(\frac{2}{7},0,0,*\right)^T$,

The remaining homogeneous control points (for $i \geq 2$) correspond to finite points. Note that the terms $*$ depend linearly on $\hat{F}$ and, in general, depend on all components of $\hat{F}$. In order to have well defined derivatives, the control points in the first row need to be basepoints. Due to Lemma 2 this is equivalent to

$$M^x_{\mathbf{i}} : \hat{F} = M^y_{\mathbf{i}} : \hat{F} = 0 \quad \text{for} \quad \mathbf{i} \in \mathbb{I}_0.$$

This system of equations can equivalently be rewritten as

$$f_{(0,0)} = f_{(0,1)} = f_{(0,2)}. \tag{4.4}$$

Theorem 2 states that $\varphi \in C^1(\bar{\Omega})$ is equivalent to $\hat{F}$ solving (4.4) and additionally

$$f_{(0,0)} + f_{(1,1)} = f_{(1,0)} + f_{(1,2)}. \tag{4.5}$$

Hence the dimension of the solution space, which is the dimension of the space $\mathcal{V}_h \cap C^1$, is 6.

Now we look at the C^2 smoothness conditions. If we apply Theorem 2 to the derivatives of φ we get that $\varphi \in C^2(\bar{\Omega})$ is equivalent to $\hat{F}$ being a solution of the system of linear equations (4.4), (4.5) and additionally

$$\begin{array}{ccccccccc} 2f_{(0,0)} & + & f_{(2,0)} & + & f_{(2,1)} & = & 2f_{(1,0)} & + & 2f_{(1,1)} \\ 2f_{(0,0)} & + & 2f_{(1,1)} & + & f_{(2,0)} & = & 4f_{(1,0)} & + & f_{(2,2)}. \end{array}$$

Hence, the dimension of the space $\mathcal{V}_h \cap C^2$ is only 4. The following example will show that this dimension depends heavily on the given geometry mapping.

EXAMPLE 2. *Consider another biquadratic patch (*$\mathbf{d} = (2,2)$*) with the control points listed in the following table:*

$\mathbf{F}_{(i,j)}$	$j=0$	$j=1$	$j=2$
$i=0$	$\left(1,0,0,f_{(0,0)}\right)^T$	$\left(1,0,0,f_{(0,1)}\right)^T$	$\left(1,0,0,f_{(0,2)}\right)^T$
$i=1$	$\left(1,\frac{1}{2},0,f_{(1,0)}\right)^T$	$\left(1,\frac{1}{2},\frac{1}{4},f_{(1,1)}\right)^T$	$\left(1,\frac{1}{2},\frac{1}{2},f_{(1,2)}\right)^T$
$i=2$	$\left(1,1,0,f_{(2,0)}\right)^T$	$\left(1,1,\frac{1}{2},f_{(2,1)}\right)^T$	$\left(1,1,1,f_{(2,2)}\right)^T$

Again, we consider arbitrary coefficients $f_{(i,j)} \in \mathbb{R}$. Note that this patch can be obtained by applying degree elevation to a degenerate bilinear patch. Figure 4 (right) shows the grid of control points. Similar to Example 1 the condition $\varphi \in C^1(\bar{\Omega})$ can be reduced to

$$\begin{array}{rcccl} f_{(0,0)} & = & f_{(0,1)} & = & f_{(0,2)} \\ f_{(1,0)} & + & f_{(1,2)} & = & 2f_{(1,1)}. \end{array} \tag{4.6}$$

The conditions characterizing $\varphi \in C^2(\bar{\Omega})$, however, do not give any additional equations in this case! Hence, the dimensions of $\mathcal{V}_h \cap C^1$ and of $\mathcal{V}_h \cap C^2$ are both equal to 6. One can even show that $\dim(\mathcal{V}_h \cap C^\infty) = 6$ for Example 2, due to the fact that the patch is actually a degree elevated bilinear patch.

4.4. *Connection between C^k smoothness and H^k regularity*

We conclude a direct connection between the presented C^k smoothness conditions and the H^k regularity conditions. In Takacs & Jüttler (2011), Takacs & Jüttler (2012) we analyzed the regularity of the isogeometric test functions in the presence of singularities in the geometry mapping. In that case some of the functions φ may not fulfill $\varphi \in H^1(\Omega)$ or $\varphi \in H^2(\Omega)$. These conditions are equivalent to the boundedness of the H^1 seminorm

$$|\varphi|^2_{H^1(\Omega)} = \int_\Omega \left(\partial_x \varphi(x,y)\right)^2 + \left(\partial_y \varphi(x,y)\right)^2 \,\mathrm{d}(x,y)$$

and the H^2 seminorm

$$|\varphi|^2_{H^2(\Omega)} = \int_\Omega \left(\partial_x^2 \varphi(x,y)\right)^2 + 2\left(\partial_x\partial_y \varphi(x,y)\right)^2 + \left(\partial_y^2 \varphi(x,y)\right)^2 \,\mathrm{d}(x,y)$$

of the isogeometric test function φ, respectively.

In the following we assume that the graph surface $\mathbf{F}$ is a polynomial patch, i.e. all weights fulfill $w_{\mathbf{i}} = 1$. In this case we know the following.

THEOREM 3 (TAKACS & JÜTTLER (2012)). *Let $\varphi \in \mathcal{V}_h$ be a function on an edge-degenerated patch with homogeneous control points $\mathbf{F}_{\mathbf{i}}$. The function φ fulfills $\varphi \in H^1(\Omega)$ if and only if*

$$\mathbf{F}_{\mathbf{i}} = \mathbf{F}_{\mathbf{j}} \;\; \forall \mathbf{i}, \mathbf{j} \in \mathbb{I}_0.$$

Furthermore, if all

$$\mathbf{F}_{\mathbf{i}}, \;\; \mathbf{i} \in \mathbb{I}_0 \cup \mathbb{I}_1 = \left\{(i,j) : 0 \le i \le 1,\, 0 \le j \le d_2\right\},$$

lie in a common plane, then $\varphi \in H^2(\Omega)$.

Proof. This is a direct consequence of Theorems 4.4 and 4.5 in Takacs & Jüttler (2012). □

If we compare these H^1 and H^2 regularity results with the C^0 and C^1 smoothness results presented earlier, we get the following.

COROLLARY 2. *Let $\varphi \in \mathcal{V}_h$ be a function on an edge-degenerated patch. Then $\varphi \in H^k(\Omega)$ if and only if $\varphi \in C^{k-1}(\bar{\Omega})$ for $k = 1, 2$.*

Proof. For $k = 1$ this follows from Theorem 3 and Proposition 2. The inclusion $\varphi \in C^1(\bar{\Omega}) \Rightarrow \varphi \in H^2(\Omega)$ follows from Theorem 3 and Proposition 2. We only give a sketch of the proof for $\varphi \in C^1(\bar{\Omega}) \Leftarrow \varphi \in H^2(\Omega)$. Assume there exists a function $\varphi \notin C^1$. Without loss of generality $\partial_x \varphi \notin C^0$, which leads to $\partial_x \varphi \notin H^1$. This is equivalent to either $\partial_x \partial_x \varphi \notin L^2$ or $\partial_y \partial_x \varphi \notin L^2$ which leads to $\varphi \notin H^2$. In order to show $\partial_x \varphi \notin C^0 \Rightarrow \partial_x \varphi \notin H^1$ it is necessary that the equivalence from Theorem 3 is also valid for general weights, which was shown in Takacs & Jüttler (2011). □

The results presented here are related to the Sobolev embedding theorems for the Sobolev spaces $H^1(\Omega) = W^{1,2}(\Omega)$ and $H^2(\Omega) = W^{2,2}(\Omega)$ (see Adams & Fournier (2003)).

5. Conclusion

In this paper we developed an analytic representation of the derivative of an isogeometric test function on a rational patch in homogeneous coordinates. Given a function which is described by a rational graph surface $\mathbf{F}$ which is obtained by adding a coordinate to an existing geometry mapping $\mathbf{G}$, we presented a simple and systematic method to compute and to analyze the partial derivatives. This approach can be applied iteratively to derivatives of higher order.

Using this algorithmic approach we specified conditions that guarantee the smoothness of isogeometric functions. We presented C^0, C^1 and C^2 smoothness results for a certain class of singularly parametrized domains and compared them with existing H^1 and H^2 regularity results. Finally we applied the presented methods to two examples on biquadratic patches, showing that the dimension of the spaces of smooth isogeometric test functions depends heavily on the properties of the geometry mapping.

The presented framework can be generalized to other types of singularities and to higher order derivatives. It may also be of interest to analyze the complexity and the computational cost of the presented algorithms for general problems.

The presented method is a useful tool to analyze smoothness properties of general isogeometric test functions. It may be of interest to apply the algorithm to other applications where a NURBS representation of the derivatives is needed. One can for instance use the NURBS representation of the derivatives to generate an approximation of the gradient of an isogeometric function. We are convinced that implementations of many applications in isogeometric analysis, where derivatives of functions are present, can benefit substantially from using the operators D^x and D^y and exploiting their properties.

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