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Robust preconditioners for PDE-constrained optimization with limited observations using isogeometric discretization

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Abstract

In [13] robust preconditioners have been developed for PDE-constrained optimization with limited observation. The key ingredience of the approach is a variational formulation of the optimality system, where the control space and the Lagrange-multiplier space coincide. This can be acchieved for elliptic state equations by using their strong form, which implies the use of suitable finite element spaces with high regularity. As a more flexible alternative to the finite element spaces used in [13], here spline spaces within the framework of isogeometric analysis are considered for discretizing the optimality system. The presented analysis of the robustness of the preconditioner relies on weaker assumptions compared to [13]. Some numerical results are presented for illustrating the theoretical results and studying the range of applicability beyond the assumptions needed for the analysis.

1 Introduction

Numerical methods for PDE-constrained optimization problems on the form

$$\min_{f,u} \left\{ \frac{1}{2} \|u - d\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|f\|^2_{L^2(\Omega)} \right\}, \quad \text{for } \alpha > 0,$$

subject to a partial differential equation (PDE) as constraint, where Ω is a Lipschitz domain, have be extensively studied and robust preconditioners (with respect to α and the mesh-size) have been developed, see for example [14, 17]. Common for these analysis of α-independent bounds, are the assumption that the observation, $\|u - d\|^2_{L^2(\Omega)}$, is available throughout the whole domain, Ω. In several practical applications the observation is often available only on parts of the domain or only on the boundary of the domain.

A method for constructing robust preconditioners for PDE-constrained optimization with limited observation was suggested in [13]. For this method, higher regularity is required for the state function, u. For elliptic PDEs, $H^2(\Omega)$-conforming discretization were required. $H^2(\Omega)$-conforming finite elements are not generally easy to construct. In [13], the authors suggested the Bogner-Fox-Schmit (BFS) element. However, the BFS elements are restricted to rectangles and for complicated domain the meshing becomes a challenge. Another possibility is the Argyris triangle. However, this element is challenging to implement because of the high number of degree of freedoms (DoFs).

Isogeometric analysis (IGA) is a numerical method for solving PDEs. It was first proposed in [11] and have - since then - attracted considerable research attention. One of the good feature of IGA, is the possibility to have discretizations with high regularity. In IGA spline spaces are used for both representing the computational domain and discretizing PDEs. Refinement can be done with respect to mesh size, polynomial degree of the splines and the regularity. We refer to the monograph [2], the survey article [3] and the references therein for an overview of the topic.

IGA's feature of having discretizations with high regularity along with the many geometries it can represent, makes it a suitable discretization technique for solving PDE-constrained optimization problems with limited observation in the setting of [13]. We discretize such a problem and present a computational efficient preconditioner, in the IGA setting, using tensor-product B-splines. Stability of the discretized problem is shown under weaker conditions than in [13]. We will also investigate the robustness of the preconditioner beyond these conditions in numerical experiments.

The remainder of this article is organized as the following. In Section 2 we present the abstract theory from [13]. In Section 3 we derive a set of conditions that ensures a stable discretization. In Section 4 we very briefly introduce tensor-product B-splines and mesh refinement. In Section 5 we apply the theory from Section 2 and Section 3 to a specified PDE-constrained optimization problem, along with a corresponding suitable preconditioner. Numerical results are giving is Section 6 and the discussion can be found in Section 7.

2 The abstract setting

In this section, we briefly present the abstract theory from [13] for solving PDE-constrained optimization with limited observation. The main idea is that the control space and the test space of the state equation, must coincide.

Problem 2.1.

$$\min_{f\in W,\, u\in U} \left\{ \frac{1}{2} \|Tu - d\|_O^2 + \frac{\alpha}{2} \|f\|_W^2 \right\}, \quad \text{for } \alpha > 0 \tag{1}$$

subject to

$$\langle Au, w\rangle + (f, w)_W = 0, \quad \forall\, w \in W. \tag{2}$$

Here $A : U \to W'$ is the state operator. U is the state space and W is the test space. $T : U \to O$ is the observation operator and O is the observation space. Equation (2) is a variational formulation of the state equation where both f and v are in the same space W. This is normally a non-standard variational formulation which requires additional regularity on the state space U. We assume W, U and O are Hilbert spaces. We have the following two assumptions on A and T.

1. $A : U \to W'$ is a continuous linear operator with closed range, where for some constant C_1, the following equation holds,

$$\inf_{\tilde{u} \in \text{Ker } A} \|u - \tilde{u}\|_U \leq C_1 \|Au\|_{W'}, \quad \forall\, u \in U. \tag{3}$$

2. $T : U \to O$ is linear, bounded and invertible on the kernel of A. That is, there exits a constant C_2 such that the following equation holds,

$$\|u\|_U \leq C_2 \|Tu\|_O, \quad \forall\, u \in \text{Ker } A. \tag{4}$$

The associated Lagrangian to Problem 2.1 is

$$\mathcal{L}\,(f, u, w) = \frac{1}{2}\, \|Tu - d\|_O^2 + \frac{\alpha}{2}\, \|f\|_W^2 + \langle Au, w \rangle + (f, w)_W,$$

with $f, w \in W$ and $u \in U$. Using the first order optimality conditions

$$\frac{\partial \mathcal{L}}{\partial f} = 0, \quad \frac{\partial \mathcal{L}}{\partial u} = 0, \quad \frac{\partial \mathcal{L}}{\partial w} = 0,$$

we obtain the KKT system:

Determine $(f, u, w) \in W \times U \times W$ for $\alpha > 0$ such that

$$\begin{aligned} \alpha\,(f, \psi)_W + (w, \psi)_W &= 0, \quad \forall\, \psi \in W, \\ (Tu - d, T\phi)_O + \langle A'w, \phi \rangle &= 0, \quad \forall\, \phi \in U, \\ (w, \xi)_W + \langle Au, \xi \rangle &= 0, \quad \forall\, \xi \in W. \end{aligned}$$

We rewrite this in a more algebraic notation:
Let the symbol “′“ denote dual operators and dual spaces and let

$$\begin{aligned} &M : W \to W', && f \mapsto (f, \cdot)_W\,, \\ &K : U \to U', && u \mapsto (Tu, T\cdot)_O\,, \\ &\tilde{K} : O \to U', && d \mapsto (d, T\cdot)_O\,. \end{aligned}$$

Problem 2.2. *Determine* $(f, u, w) \in W \times U \times W$ *for* $\alpha > 0$ *such that*

$$\underbrace{\begin{pmatrix} \alpha M & & M \\ & K & A' \\ M & A & 0 \end{pmatrix}}_{\mathcal{A}_\alpha} \begin{pmatrix} f \\ u \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{K}_\partial d \\ 0 \end{pmatrix}. \tag{5}$$

We define appropriate scaled norms, which we use later. Let $f, w \in W$ and $u \in U$.

$$\|f\|_{W_\alpha}^2 = \alpha\, \|f\|_W^2\,, \tag{6}$$

$$\|u\|_{U_\alpha}^2 = \alpha\, \|Au\|_{W'}^2 + \|Tu\|_O^2\,, \tag{7}$$

$$\|w\|_{W_{\alpha^{-1}}}^2 = \frac{1}{\alpha}\, \|w\|_W^2\,. \tag{8}$$

It can be shown that $\|\cdot\|_{U_\alpha}^2$ is, in fact, a norm on U given the two assumption on A and T hold, see [13]. Set $\mathcal{V} = W_\alpha \times U_\alpha \times W_{\alpha^{-1}}$ and let $\mathcal{A}_\alpha$ be the operator in Equation (5).

Theorem 2.1. *The operator $\mathcal{A}_\alpha : \mathcal{V} \to \mathcal{V}'$ is bounded and continuously invertible for $\alpha > 0$. There exist positive constants $\underline{c}$ and $\bar{c}$ that are independent of α such that*

$$\underline{c} \leq \sup_{0 \neq y \in \mathcal{V}} \frac{\langle \mathcal{A}_\alpha x, y\rangle}{\|y\|_{\mathcal{V}} \|x\|_{\mathcal{V}}} \leq \bar{c}, \quad \forall\, x \in \mathcal{V}, \text{ where } x \neq 0. \tag{9}$$

Proof. The proof can be found in [13], where they show the four Brezzi conditions of Brezzi's theory for saddle point problems [1]. □

Now that it is shown that $\mathcal{A}_\alpha$ is a isomorphism, we present a suitable preconditioner,

$$\mathcal{B}_\alpha = \begin{bmatrix} \alpha M & & \\ & \alpha A' M^{-1} A + K & \\ & & \frac{1}{\alpha} M \end{bmatrix}^{-1}. \tag{10}$$

Theorem 2.2. *If the following assumptions hold,*

1. *M is self-adjoint and positive definite,*
2. *$A' M^{-1} A + K$ is positive definite,*
3. *K is self-adjoint and positive semi-definite,*

then the spectral condition number of $\mathcal{B}_\alpha \mathcal{A}_\alpha$ is bounded independently of α.

$$\kappa\left(\mathcal{B}_\alpha \mathcal{A}_\alpha\right) \approx 4.089. \tag{11}$$

Proof. The proof can be found in Section 7 in [13]. □

3 Discretization conditions

To check that one has a suitable discretization, one can check the Brezzi conditions for the discretized spaces. However, the coercivity condition is not trivial to verify. In [13], they give a sufficient, but not necessary condition for their discrete problem. We will give more relaxed conditions on the discretization of the spaces by applying the theory from [19] on the continuous problem. In one way, we are reproving Theorem 2.1, however, the goal is to get a different set of conditions, which are easier to work with on the discrete level. First we present the main result from [19].

Let V and Q be Hilbert spaces, let $X = V \times Q$. We have the following bounded operators $\bar{A} \in L(V, V')$, $B \in L(V, Q')$ and $C \in L(Q, Q')$, where $\bar{A}$ and C are self-adjoint and semi-definite. Let $\bar{f}$ and g be bounded linear functionals on V and Q, respectively. We have the following problem formulation:

Problem 3.1. *Find $z = (\bar{u}, p) \in X$ such that*

$$\mathcal{A}z = \begin{pmatrix} \bar{A} & B' \\ B & -C \end{pmatrix} \begin{pmatrix} \bar{u} \\ p \end{pmatrix} = \begin{pmatrix} \bar{f} \\ g \end{pmatrix} \tag{12}$$

holds.

Let $\mathcal{I}_H : H \to H'$ be given by

$$\langle \mathcal{I}_H x, y \rangle_H = (x, y)_H \quad \text{for } x, y \in H,$$

where H is a Hilbert space and $\langle \cdot, \cdot \rangle_H$ is the duality pairing on H. $\mathcal{I}_H$ is an isometric isomorphism between H and its dual space H'. The inverse, $\mathcal{I}_H^{-1}$, is called the Riesz-isomorphism.

The main result in [19] is stated in the following theorem.

Theorem 3.1. *If there are constants $\underline{\gamma}_v$, $\bar{\gamma}_v$, $\underline{\gamma}_q$, $\bar{\gamma}_q > 0$ such that*

$$\underline{\gamma}_v \langle \mathcal{I}_V \bar{w}, \bar{w} \rangle \le \left\langle \left(\bar{A} + B' \mathcal{I}_Q^{-1} B \right) \bar{w}, \bar{w} \right\rangle \le \bar{\gamma}_v \langle \mathcal{I}_V \bar{w}, \bar{w} \rangle, \quad \forall \, \bar{w} \in V, \tag{13}$$

$$\underline{\gamma}_q \langle \mathcal{I}_Q r, r \rangle \le \left\langle \left(C + B \mathcal{I}_V^{-1} B' \right) r, r \right\rangle \le \bar{\gamma}_q \langle \mathcal{I}_Q r, r \rangle, \quad \forall \, \bar{r} \in Q, \tag{14}$$

then

$$\underline{c}_x \langle \mathcal{I}_X z, z \rangle \le \langle \mathcal{A} z, z \rangle \le \bar{c}_x \langle \mathcal{I}_X z, z \rangle, \quad \forall \, \bar{z} \in X, \tag{15}$$

is satisfied with constants $\underline{c}_x$, $\bar{c}_x > 0$ that depend only on $\underline{\gamma}_v$, $\bar{\gamma}_v$, $\underline{\gamma}_q$ and $\bar{\gamma}_q$. Also, vice versa, if Equation (15) hold with constants $\underline{c}_x$, $\bar{c}_x > 0$, then Equation (13) and Equation (14) hold with constants $\underline{\gamma}_v$, $\bar{\gamma}_v$, $\underline{\gamma}_q$, $\bar{\gamma}_q > 0$, dependent only on $\underline{c}_x$ and $\bar{c}_x$.

Proof. See Theorem 2.6 in [19]. □

If we can find $\mathcal{I}_V$ and $\mathcal{I}_Q$ that satisfy Condition (13) and Condition (14), we obtain suitable norms and a robust preconditioner. The two equations can be rewritten as

$$\mathcal{I}_V \sim \bar{A} + B' \mathcal{I}_Q^{-1} B \quad \text{and} \quad \mathcal{I}_Q \sim C + B \mathcal{I}_V^{-1} B'. \tag{16}$$

Writing Problem 2.2 in the setting of Problem 3.1 and Theorem 3.1, we see that

$$\bar{A} = \begin{pmatrix} \alpha M & 0 \\ 0 & K \end{pmatrix}, \quad B' = \begin{pmatrix} M \\ A' \end{pmatrix}$$

and $C = 0$. Furthermore $\bar{u} = (f, u)$ and $p = w$. The spaces translate to $V = W \times U$ and $Q = W$. And finally $\bar{f} = (0, \tilde{K}_\partial d)$ and $g = 0$.

Following the ideas of [13], we choose $\mathcal{I}_Q = \frac{1}{\alpha} M$. Having chosen an $\mathcal{I}_Q$, we plug this into the first condition in (16) and get

$$\mathcal{I}_V \sim \begin{pmatrix} 2\alpha M & \alpha A \\ \alpha A' & K + \alpha A' M^{-1} A \end{pmatrix}.$$

This is, however, not easy to invert and we would like to write $\mathcal{I}_V$ on the diagonal form

$$\mathcal{I}_V \sim \begin{pmatrix} \alpha M & 0 \\ 0 & K + \alpha A' M^{-1} A \end{pmatrix}.$$

To do this, we use the following lemma.

Lemma 3.1. *Let $\mathcal{M}$ be a bounded self-adjoint positive operator, $\mathcal{M} : H_1 \times H_2 \to H_1' \times H_2'$, of the form*

$$\mathcal{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \tag{17}$$

where H_1 and H_2 are Hilbert spaces, and let $\mathcal{D}$ be an operator on the form

$$\mathcal{D} = \begin{pmatrix} D_{11} & \\ & D_{22} \end{pmatrix}, \tag{18}$$

where $D_{11} : H_1 \to H_1'$ and $D_{22} : H_2 \to H_2'$ are bounded self-adjoint positive operators. Then $\mathcal{M}$ and $\mathcal{D}$ are equivalent, $\mathcal{M} \sim \mathcal{D}$, if

$$M_{11} \sim D_{11}, \quad M_{22} \sim D_{22}, \quad \text{and} \quad M_{11} \leq C_M \left(M_{11} - M_{12} M_{22}^{-1} M_{21} \right), \tag{19}$$

for a constant, $C_M > 1$.

Proof. First we show that $M_{11} \leq C_M \left(M_{11} - M_{12} M_{22}^{-1} M_{21} \right)$ leads to the strengthened Cauchy-Schwarz inequality. Let $C_M = \frac{1}{1-\gamma^2}$, where $\gamma < 1$. The inequality can be written as

$$M_{12} M_{22}^{-1} M_{21} \leq \gamma^2 M_{11},$$

which is equivalent to

$$\left\langle M_{12} M_{22}^{-1} M_{21} x, x \right\rangle = \sup_{y \in H_2, y \neq 0} \frac{\langle M_{21} x, y \rangle^2}{\langle M_{22} y, y \rangle} \leq \gamma^2 \langle M_{11} x, x \rangle.$$

From this we get the strengthened Cauchy-Schwarz inequality

$$| \langle M_{21} x, y \rangle | \leq \gamma \sqrt{\langle M_{11} x, x \rangle} \sqrt{\langle M_{22} y, y \rangle} \quad \forall\, x \in H_1, y \in H_2. \tag{20}$$

We now show the equivalency

$$\underline{\beta} \langle \mathcal{D} z, z \rangle \leq \langle \mathcal{M} z, z \rangle \leq \bar{\beta} \langle \mathcal{D} z, z \rangle \quad \forall\, z \in H_1 \times H_2,$$

where $\underline{\beta}$ and $\bar{\beta}$ are constants. Let $z = (x, y)$, then by using Inequality (20), we get

$$\begin{aligned}\langle \mathcal{M}z, z\rangle &= \langle M_{11}x, x\rangle + 2\langle M_{21}x, y\rangle + \langle M_{22}y, y\rangle \\ &\leq \langle M_{11}x, x\rangle + 2\gamma\sqrt{\langle M_{11}x, x\rangle}\sqrt{\langle M_{22}y, y\rangle} + \langle M_{22}y, y\rangle \\ &\leq (1+\gamma)\langle M_{11}x, x\rangle + (1+\gamma)\langle M_{22}y, y\rangle \\ &\leq (1+\gamma)\bar{\beta}_1\langle D_{11}x, x\rangle + (1+\gamma)\bar{\beta}_1\langle D_{22}y, y\rangle \\ &\leq (1+\gamma)\max\{\bar{\beta}_1, \bar{\beta}_2\}\langle \mathcal{D}z, z\rangle,\end{aligned}$$

where $\bar{\beta}_1$ and $\bar{\beta}_2$ are the upper constants for the equivalency $M_{11} \sim D_{11}$ and $M_{22} \sim D_{22}$, respectively. Similarily, one can show the lower bound

$$(1-\gamma)\min\{\underline{\beta}_1, \underline{\beta}_2\}\langle \mathcal{D}z, z\rangle \leq \langle \mathcal{M}z, z\rangle.$$

□

Theorem 3.2. *If the following condition holds:*

1. *M is bounded, self-adjoint and positive definite,*
2. *$K + \alpha A' M^{-1} A$ is bounded and positive definite,*
3. *K is self-adjoint and positive semi-definite,*

then $\mathcal{I}_V^1$ and $\mathcal{I}_V^2$ are equivalent, where

$$\mathcal{I}_V^1 = \begin{pmatrix} 2\alpha M & \alpha A \\ \alpha A' & K + \alpha A' M^{-1} A \end{pmatrix} \quad \text{and} \quad \mathcal{I}_V^2 = \begin{pmatrix} \alpha M & 0 \\ 0 & K + \alpha A' M^{-1} A \end{pmatrix}.$$

Furthermore Equation (13) and (14) in Theorem 3.1 hold for $\mathcal{I}_V^2$ and $\mathcal{I}_Q = \frac{1}{\alpha}M$.

Proof. From Lemma 3.1, we know that $\mathcal{I}_V^1$ and $\mathcal{I}_V^2$ are equivalent if

$$M \leq \left(2M - \alpha A\left(K + \alpha A' M^{-1} A\right)^{-1} A'\right). \tag{21}$$

To show the inequality above, it is sufficient to show

$$M \geq \alpha A\left(K + \alpha A' M^{-1} A\right)^{-1} A'. \tag{22}$$

We define $A_\alpha := \sqrt{\alpha}A$ and $B_1 := K + \alpha A' M^{-1} A$. From the conditions, one can see that B_1 is bounded, self-adjoint and positive definite. Hence, there exist an unique $B_1^{-1/2}$ such that $B_1^{-1} = B_1^{-1/2} B_1^{-1/2}$, see [16]. The same is true for M, $M^{-1} = M^{-1/2} M^{-1/2}$. Inequality (22) can be written as

$$M \geq A_\alpha B_1^{-1/2} B_1^{-1/2} A_\alpha'.$$

We multiply $M^{-1/2}$ on both sides of the inequality above

$$I \geq M^{-1/2} A_\alpha B_1^{-1/2} B_1^{-1/2} A'_\alpha M^{-1/2} = D'D,$$

where $D = B_1^{-1/2} A'_\alpha M^{-1/2}$. We have

$$I \geq D'D \quad \Leftrightarrow \quad I \geq DD'.$$

and

$$\begin{aligned} I \geq DD' &= B_1^{-1/2} A'_\alpha M^{-1/2} M^{-1/2} A_\alpha B_1^{-1/2}, \\ &= B_1^{-1/2} A'_\alpha M^{-1} A_\alpha B_1^{-1/2}. \end{aligned}$$

We multiply the last inequality with $B_1 B_1^{-1/2}$ from the left, and with $B_1^{-1/2} B_1$ from the right,

$$B_1 B_1^{-1/2} I B_1^{-1/2} B_1 = B_1 \geq A'_\alpha M^{-1} A_\alpha.$$

Recalling our definition of A_α and B_1, we see that the inequality above states,

$$K + \alpha A' M^{-1} A \geq \alpha A' M^{-1} A.$$

which is true, since K is positive semi-definite. Hence, Inequality (21) holds. The next step is to show that Condition (16) holds for $\mathcal{I}_V^2$ and $\mathcal{I}_Q = \frac{1}{\alpha} M$. The second condition is already shown and we are left with checking the first condition.

$$\begin{aligned} B \mathcal{I}_V^{-1} B' &= (M, A) \begin{pmatrix} \frac{1}{\alpha} M^{-1} & 0 \\ 0 & (K + \alpha A' M^{-1} A)^{-1} \end{pmatrix} \begin{pmatrix} M \\ A' \end{pmatrix} \\ &= \frac{1}{\alpha} M + A \left(K + \alpha A' M^{-1} A \right)^{-1} A'. \end{aligned}$$

To see that $\frac{1}{\alpha} M$ is equivalent to $\frac{1}{\alpha} M + A \left(K + \alpha A' M^{-1} A \right)^{-1} A'$, we see that

$$\frac{1}{\alpha} M \leq \frac{1}{\alpha} M + A \left(K + \alpha A' M^{-1} A \right)^{-1} A'$$

is true. The other direction is already shown earlier in the proof. □

From $\mathcal{I}_V^2$ and $\mathcal{I}_Q$ from Theorem 3.2 we obtain the norms

$$\|f\|^2_{W_\alpha} = \alpha \|f\|^2_W, \tag{23}$$

$$\|u\|^2_{U_\alpha} = \alpha \|Au\|^2_{W'} + \|Tu\|^2_O, \tag{24}$$

$$\|w\|^2_{W_{\alpha^{-1}}} = \frac{1}{\alpha} \|w\|^2_W \tag{25}$$

and the preconditioner

$$\mathcal{B}_\alpha = \begin{bmatrix} \alpha M & & \\ & K + \alpha A' M^{-1} A & \\ & & \frac{1}{\alpha} M \end{bmatrix}^{-1}, \tag{26}$$

which is robust with respect to α.

The conditions in Theorem 3.2 and the preconditioner above are identical to Theorem 2.2, which is proven in [13]. We have in this section shown that the conditions also are sufficient for ensuring a stable discretization. Thus having a different set of conditions, than the Brezzi conditions, which are easier to work with on the discrete level.

Lemma 3.2. *If condition 1 and 3 in Theorem 3.2 are satisfied, then condition 2 is equivalent to; A is bounded and*

$$ker\,(K) \cap ker\,(A) = \{0\}. \tag{27}$$

Proof. Condition 1 ensures that $A'M^{-1}A$ is self-adjoint and semi-definite,

$$\left\langle A'M^{-1}Au, u\right\rangle = \left\langle M^{-1}Au, Au\right\rangle = \left\langle M^{-\frac{1}{2}}Au, M^{-\frac{1}{2}}Au\right\rangle \geq 0.$$

From condition 3 we have that K is bounded and positive semi-definite. It remains to show that $\langle \left(K + \alpha A'M^{-1}A\right) u, u\rangle$ is zero if and only if $u = 0$, which is equivalent to $\ker\,(K) \cap \ker\,(A) = \{0\}$.

□

4 IGA discretization

The variational formulation in Equation (2) typically requires more smoothness on the state space. For elliptic PDE constraints, which we will consider, the state space must be in $H^2\,(\Omega)$. Finding suitable $H^2\,(\Omega)$-conforming discretizations using the finite element method is not trivial. In [13], they use the Bogner-Fox-Schmit element. However, this element requires a rectangular mesh. The Argyris triangle could be considered, however, the implementation of this element is also a challenge. IGA discretization has the advantage of creating discretizations with high smoothness and a lot of geometries can be represented exactly through B-splines and NURBS[1]. In this section, we give a brief introduction to splines and IGA. For more information on splines, see [4], and for more information about IGA, see [2, 3].

[1]For simplicity, non-uniform rational basis spline (NURBS) will not be addressed in this paper. We refer to [15, 3] for further information about NURBS and IGA.

4.1 Univariate B-splines

Let p and n be two positive integers, we use the p-open knot vector $\Xi = \{\xi_1, \ldots, \xi_{n+p+1}\}$, where

$$0 = \xi_1 = \ldots = \xi_{p+1} < \xi_{p+2} \leq \ldots \leq \xi_n < \xi_{n+1} = \ldots = \xi_{n+p+1} = 1.$$

We define the B-spline basis function through the recursive Cox-de Boor formula. For p = 0,

$$B_i^0(\xi) = \begin{cases} 1, & \text{if } \xi_i \leq \xi < \xi_{i+1}, \\ 0, & \text{otherwise.} \end{cases}$$

For $p > 0$,

$$B_i^p(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} B_i^{p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} B_{i+1}^{p-1}(\xi),$$

where it is formally assumed that $0/0 = 0$. From the definition above, we can view p as the polynomial order of the spline. Let us introduce the knot multiplicity vector $\mathbf{r} = \{r_1, \ldots, r_m\}$, where m is the number of distinguishable knots in Ξ. That is; r_i is defined to be the multiplicity of the i-th distinguished knot. We assume that $r_i \leq p + 1$. The i-th B-spline basis function has $\alpha_i := p - r_i$ continuous derivatives. We define the regularity vector[2] $\boldsymbol{\alpha} := \{\alpha_1, \ldots, \alpha_m\}$. From our definition of the p-open knot vector, $\alpha_1 = \alpha_m = -1$. The space spanned by the basis functions B_i^p from the knot vector Ξ, is denoted by

$$\hat{S}_{\boldsymbol{\alpha}}^p := \text{span}\{B_i^p\}_{i=1}^n.$$

A spline curve is a linear combination of B-splines basis functions and control points, $\{\mathbf{c}_i\}_{i=1}^n$,

$$C(\xi) = \sum_{i=1}^n \mathbf{c}_i B_i^p(\xi), \quad \mathbf{c}_i \in \mathbb{R}^d,$$

where d is a positive integer.

4.2 Multivariate B-splines

Multivariate B-splines can easily be defined through the tensor-product construction of univariate B-splines. Given the positive integers p_l and n_l, for $l = 1, \ldots, d$, let $\Xi_l = \{\xi_{1,l}, \ldots, \xi_{n+p+1,l}\}$ be p-open knot vectors with the associated regularity vectors $\boldsymbol{\alpha}_l$. For each knot vector, Ξ_l, and degree, p_l, we have the associated univariate B-spline basis functions, $B_{i_l,l}^{p_l}$, for $i_l = 1, \ldots, n_l$. We define the tensor product B-spline basis functions as

$$B_{i_1,\ldots,i_d}^{p_1,\ldots,p_d}(\xi_1, \ldots, \xi_d) := B_{i_1,1}^{p_1}(\xi_1) \otimes \ldots \otimes B_{i_d,d}^{p_d}(\xi_d),$$

[2]When a integer, say 1, is used instead of a vector $\boldsymbol{\alpha}$. It means that every element in $\boldsymbol{\alpha}$ is equal to 1. Except the first and last, which are equal to -1.

for $i_1 = 1, \dots, n_1, \ \dots, \ i_d = 1, \dots, n_d$, where $(\xi_1, \dots, \xi_d) \in (0,1)^d$. This is will be refered to as the parametric domain, $(0,1)^d = \hat{\Omega} \subset \mathbb{R}^d$. The tensor-product B-spline space is defined as

$$\hat{S}^{p_1,\dots,p_d}_{\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_d} := \operatorname{span}\left\{B^{p_1,\dots,p_d}_{i_1,\dots,i_d}\right\}^{n_1,\dots,n_d}_{i_1=1,\dots,i_d=1}. \tag{28}$$

We can define a spline surface or spline volume as a linear combination of B-spline basis functions and control points, $\{\mathbf{c}_{i_1,\dots,i_d}\}^{n_1,\dots,n_d}_{i_1=1,\dots,i_d=1}$,

$$\mathbf{F}(\xi_1, \dots, \xi_d) = \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} \mathbf{c}_{i_1,\dots,i_d} B^{p_1}_{i_1,1}(\xi_1) \dots B^{p_d}_{i_d,d}(\xi_d), \quad \mathbf{c}_{i_1,\dots,i_d} \in \mathbb{R}^d. \tag{29}$$

The domain which $\mathbf{F}$ maps to, will be refered to as the physical domain and we denote it as $\Omega \subset \mathbb{R}^d$. I general, the dimension of the parametric domain and the dimension of the physical domain are not the same. However, we will only consider the case where they are equal. We call $\mathbf{F}$ a geometry map. We assume (unless stated otherwise) that our physical domain can be parametrized exactly by a geometry mapping. Furthermore, we assume that $\mathbf{F}$ has a piece-wise smooth inverse.

4.3 Mesh and h-refinement

Given the knot vectors $\Xi_l = \{\xi_{1,l}, \dots, \xi_{n+p+1,l}\}$, for $l = 1, \dots, d$, we define a mesh on the parametric domain $\hat{\Omega}$,

$$\mathcal{M}_h = \{Q = \otimes_{l=1,\dots,d}(\xi_{i_l,l}, \xi_{i_l+1,l}), 1 \le i_l \le m_l - 1\}.$$

Using the geometry mapping, we get a mesh on the physical domain,

$$\mathcal{K}_h = \{K : K = \mathbf{F}(Q) : Q \in \mathcal{M}_h\}.$$

We uniformly h-refine the mesh by inserting a knot, $\frac{1}{2}(\xi_{i_l+1,l} + \xi_{i_l,l})$, for every distinguishable knots $\xi_{i_l,l}$ and $\xi_{i_l+1,l}$.

One of the main ideas of IGA is to use the same spline space for representing the geometry and the discretization of the PDE. That is, we use the Galerkin method with the following space, as discretization space,

$$S^{p_1,\dots,p_d}_{\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_d} := \left\{f \circ \mathbf{F}^{-1} : f \in \hat{S}^{p_1,\dots,p_d}_{\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_d}\right\}. \tag{30}$$

5 Discretizing with an elliptic PDE-constraint

5.1 The continuous problem

In Section 2 we presented the general theory of PDE-constrained optimization with limited observation from [13]. A brief introduction to IGA was given in Section 4 and the conditions for a stable discretization was given in Theorem 3.2. In this section we choose a specific optimal control problem with an elliptic PDE-constraint, which we will discretize and solve.

Definition 5.1.

We define the space

$$\bar{H}^2 := \left\{ u \in H^2(\Omega) \,|\, u = 0 \text{ on } \partial\Omega \right\},$$

where $\Omega \subset \mathbb{R}^2$ *with the inner product*

$$(u,v)_{\bar{H}^2} := (\Delta u, \Delta v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)} + (u,v)_{L^2(\Omega)}.$$

Problem 5.1.

$$\min_{f \in L^2(\Omega),\, u \in \bar{H}^2} \left\{ \frac{1}{2} \left\| \frac{\partial u}{\partial \mathbf{n}} - d_n \right\|^2_{L^2(\partial\Omega)} + \frac{\alpha}{2} \|f\|^2_{L^2(\Omega)} \right\}, \quad \text{for } \alpha > 0 \tag{31}$$

subject to

$$(-\Delta u, w)_{L^2(\Omega)} + (f, w)_{L^2(\Omega)} = 0, \quad \forall\, w \in L^2(\Omega). \tag{32}$$

Remark 5.1.

Note that (32) is a non-standard variational formulation of a Poisson equation with homogeneous Dirichlet boundary condition. The theory in Section 2 requires that the control, f, and the adjoit state, w, are in the same space, $L^2(\Omega)$. Hence, integration by parts is not used in this variational formulation. The space for u then becomes $\bar{H}^2$ and not the standard $H^1_0(\Omega)$. Our desired state for the normal derivative of u, is $d_n \in L^2(\partial\Omega)$.

We now check if Problem 5.1 fulfills the assumptions in Section 2. First we note the $\bar{H}^2$, $L^2(\Omega)$ and $L^2(\partial\Omega)$ are Hilbert spaces. The operator $A = -\Delta : \bar{H}^2 \to L^2(\Omega)$ is linear and continuous. We need show that it has a closed range; that is, we need to show

$$\|u\|_{\bar{H}^2} \le \|Au\|_{L^2(\Omega)} \quad \forall\, u \in \bar{H}^2 / \operatorname{Ker} A. \tag{33}$$

Theorem 5.1. *Assuming Ω is a bounded polygonal open subset of $\mathbb{R}^2$, then*

$$\|u\|_{H^2(\Omega)} \le C_r \|\Delta u\|_{L^2(\Omega)}, \quad u \in \bar{H}^2, \tag{34}$$

for some constant C_r.

Proof. Can be found in [7]. □

From Theorem 5.1 and the fact that the $H^2(\Omega)$ norm and the $\bar{H}^2$ norm are equivalent for $u \in \bar{H}^2$, it follows that Inequality (33) holds. The operator $T = \frac{\partial u}{\partial \mathbf{n}} : \bar{H}^2 \to L^2(\partial\Omega)$ is linear and bounded. From Theorem 5.1, it follows that the kernel of A is trivial and hence, T is invertible on the kernel of A.

In summary, the theory from Section 2 applies to Problem 5.1, in particular Theorem 2.1. We write up the optimality system from Problem 5.1.

Problem 5.2. *Determine* $(f, u, w) \in L^2(\Omega) \times \bar{H}^2 \times L^2(\Omega)$, *for* $\alpha > 0$ *such that*

$$\begin{pmatrix} \alpha M & & M \\ & K & A^T \\ M & A & 0 \end{pmatrix} \begin{pmatrix} f \\ u \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{M}_\partial d_n \\ 0 \end{pmatrix}, \tag{35}$$

where,

$$\begin{aligned} M &: L^2(\Omega) \to \left(L^2(\Omega)\right)', & f &\mapsto (f, \cdot)_{L^2(\Omega)}, \\ K &: \bar{H}^2 \to \left(\bar{H}^2\right)', & u &\mapsto \left(\frac{\partial u}{\partial \mathbf{n}}, \frac{\partial \cdot}{\partial \mathbf{n}}\right)_{L^2(\partial\Omega)}, \\ A &: \bar{H}^2 \to \left(L^2(\Omega)\right)', & u &\mapsto (-\Delta u, \cdot)_{L^2(\Omega)}, \\ \tilde{M}_\partial &: L^2(\partial\Omega) \to \left(\bar{H}^2\right)', & d_n &\mapsto \left(d_n, \frac{\partial \cdot}{\partial \mathbf{n}}\right)_{L^2(\partial\Omega)}. \end{aligned}$$

We use the following parameter dependent norms,

$$\|f\|^2_{L^2_\alpha(\Omega)} = \alpha \|f\|^2_{L^2(\Omega)}, \tag{36}$$

$$\|u\|^2_{\bar{H}^2_\alpha} = \alpha \|u\|^2_{\bar{H}^2} + \left\|\frac{\partial u}{\partial \mathbf{n}}\right\|^2_{L^2(\partial\Omega)}, \tag{37}$$

$$\|w\|^2_{L^2_{\alpha^{-1}}(\Omega)} = \frac{1}{\alpha} \|w\|^2_{L^2(\Omega)}. \tag{38}$$

Note that the $\bar{H}^2_\alpha$ norm is not that same as the U_α norm from Section 2. However, they are equivalent.

5.2 The discrete problem

Problem 5.3. *Determine* $(f_h, u_h, w_h) \in W_h \times U_h \times W_h$, *for* $\alpha > 0$ *such that*

$$\begin{pmatrix} \alpha M_h & & M_h \\ & K_h & A_h^T \\ M_h & A_h & 0 \end{pmatrix} \begin{pmatrix} f_h \\ u_h \\ w_h \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{M}_{\partial,h} d_n \\ 0 \end{pmatrix}, \tag{39}$$

where M_h *is the mass matrix,* K_h *is the matrix arising from discretizing* $\left(\frac{\partial u_h}{\partial \mathbf{n}}, \frac{\partial \phi_h}{\partial \mathbf{n}}\right)_{L^2(\partial\Omega)}$, $\tilde{M}_{\partial,h} d_n$ *arises from the discretization of* $\left(d_n, \frac{\partial \phi_h}{\partial \mathbf{n}}\right)_{L^2(\partial\Omega)}$ *and* A_h *arises from the discretization of* $(-\Delta u_h, \xi_h)_{L^2(\Omega)}$, *where* $\phi_h \in U_h$ *and* $\xi_h \in W_h$. *We will use conforming discretizations; that is,* $W_h \subset L^2(\Omega)$ *and* $U_h \in \bar{H}^2$.

The conditions in Theorem 3.2 need to be satisfied in order to have a stable discretization. M_h is symmetric and positive definite and K_h is symmetric and positive semi-definite. Hence, the first and third condition are satisfied and from Lemma 3.2, we know that it remains to choose U_h and W_h such that $\ker(K_h) \cap \ker(A_h) = \{0\}$.

Let U_h be the tensor product B-spline space $\tilde{S}^{p,p}_{p-1,p-1}$, where $p \geq 2$. The $\tilde{}$ symbol means that the degrees of freedom on the boundary are prescribed by interpolation or by a L^2-projection, such that $u_h = 0$ on Γ. It follows that $K_h u_h = 0$ implies $\frac{\partial u_h}{\partial \mathbf{n}} = 0$ on Γ. We can for example choose $W_h = S^{p,p}_{p-1,p-1}$; that is, the same as U_h without the prescribed boundary conditions. Then $(-\Delta u_h, \xi_h)_{L^2(\Omega)} = (\nabla u_h, \nabla \xi_h)_{L^2(\Omega)}$, since $\frac{\partial u_h}{\partial \mathbf{n}} = 0$ on Γ and $\ker(K_h) \cap \ker(A_h) = \{0\}$. Even though our discretization space satisfies the conditions in Theorem 3.2, we have to choose U_h and W_h such that the preconditioner can be easily constructed and evaluated.

5.3 Preconditioner

We use the general preconditioner (10), presented in [13]. For the specific Problem 5.3, it becomes

$$\mathcal{B}_\alpha = \begin{bmatrix} \alpha M_h & & \\ & \alpha A_h^T M_h^{-1} A_h + K_h & \\ & & \frac{1}{\alpha} M_h \end{bmatrix}^{-1}. \tag{40}$$

We need to efficiently invert the components of the preconditioner. The inverse of the discrete mass matrix, M_h, is ill-conditioned for high spline degrees. However, it is possible to evaluate it efficiently by exploiting the tensor product structure of the splines. This method is described in [6]. We are left with efficiently inverting the matrix $\alpha A_h^T M_h^{-1} A_h + K_h$. Since the $A_h^T M_h^{-1} A_h$ matrix is dense and computationally expensive to construct, we wish to replace it with the discretized biharmonic operator,

$$\langle \tilde{A} u_h, \phi_h \rangle = (\Delta u_h, \Delta \phi_h)_{L^2(\Omega)}, \text{ where } u_h, \phi_h \in U_h. \tag{41}$$

It can be shown that $A_h^T M_h^{-1} A_h$ is spectrally equivalent to $\tilde{A}_h$ if the following inf-sup condition holds,

$$\inf_{u_h \in U_h} \sup_{v_h \in W_h} \frac{(-\Delta u_h, v_h)_{L^2(\Omega)}}{\|v_h\|_{L^2(\Omega)} \|\Delta u_h\|_{L^2(\Omega)}} \geq \beta, \tag{42}$$

for some constant $\beta > 0$. In order to efficiently precondition and solve Problem 5.3, we need to choose U_h and W_h such that Equation (42) holds. Then we can construct the spectral equivalent matrix, $\alpha \tilde{A}_h + K_h$. This matrix is inverted with a standard V-cycle of a geometric multigrid method. For information about multigrid methods, see [18].

Equation (42) can be shown easily if $\Delta U_h \subset V_h$ (set v_h equal to $-\Delta u_h$). For other chooses of U_h and W_h, we have been unable to show Equation (42). However, in Section 6 we present some numerical results that indicate that the matrices are equivalent for certain chooses of U_h and W_h.

$h \setminus p$	2	3	4	5	6
3	7	8	15	27	49
4	9	8	16	29	51
5	10	8	16	28	52
6	10	8	16	29	41
7	10	8	16	29	52

Table 1: Number of iterations. Ω is the unit square.

6 Numerical results

6.1 The biharmonic equation

The vital component of preconditioning and solving Problem 5.3 is the preconditioning of the biharmonic Equation (41). In this subsection we present some numerical results for different preconditioning techniques of the biharmonic equation in the IGA setting.

Problem 6.1. *Let $\Omega \subset \mathbb{R}^2$, find u such that*

$$\begin{aligned} \Delta^2 u &= f, \quad && \text{in} \quad \Omega, \\ u &= g_1, \quad && \text{on} \quad \partial\Omega, \\ \Delta u &= g_2, \quad && \text{on} \quad \partial\Omega. \end{aligned}$$

Problem 6.1 is written in a variational formulation corresponding to Equation (41) and we use the tensor-product B-spline space, $\tilde{S}^{p,p}_{\boldsymbol{\alpha},\boldsymbol{\alpha}}$, with $p \geq 2$ and maximum continuity, as the discretization space. The method of manufactured solutions is used and we choose the solution, $u\left(x,y\right) = \left(\left(\cos 4\pi x - 1\right)\left(\cos 4\pi y - 1\right) - x^3 - y^3\right)$, and calculate f, g_1 and g_2 from Problem 6.1. Note the essential boundary condition, g_2, is prescribed in the space U_h by projecting g_2 onto the spline space. The resulting linear system of equation, $A\mathbf{x} = \mathbf{f}$, is solve using the conjugated gradient method with a V-cycle of geometric multigrid with two pre and two post symmetric Gauss-Seidel smoothing steps. The process is stopped when

$$\frac{\|\mathbf{f} - A\mathbf{x}_k\|}{\|\mathbf{f}\|} < \epsilon, \tag{43}$$

where the threshold is set to $\epsilon = 10^{-12}$. We use a randomized normed initial vector. All computations are done with the C++ library G+Smo [9].

In the first example, Ω is the unit square. Table 1 shows the number of iterations needed to reach the desired threshold. Each column represents a polynomial order p and each row represents the number of uniform h-refinements. Table 1 indicates h-independent iteration numbers and p-dependence. In fact, there is also a dimension dependence. When the polynomial order and the dimension increases, the number of iterations increases drastically, see [5, 8]. A p-robust multigrid method for elliptic problems which are

$h \backslash p$	2	3	4	5	6
3	14	15	22	39	64
4	20	19	28	45	76
5	28	23	34	54	85
6	36	26	35	60	99
7	43	30	39	63	105

Table 2: Number of iterations when using normal Gauss-Seidel. Ω is a quarter annulus.

$h \backslash p$	2	3	4	5	6
3	6	10	16	26	41
4	8	11	17	27	41
5	8	11	17	26	40
6	9	11	17	26	41
7	9	11	17	27	41

Table 3: Number of iterations when using block Gauss-Seidel. Ω is a quarter annulus.

discretized using IGA, has recently been developed in [10]. To the authors knowledge, no such method has been developed for fourth order problems yet.

Now we look at a more complected geometry, we let Ω be a quarter of an annulus[3], where the inner radius is 1 and the outer radius is 2. Again we use two pre and two post symmetric Gauss-Seidel smoothing steps. Table 2 shows the iteration numbers. The table shows that the number of iterations are no longer h-independent, but rather a logarithmic dependents on h. It is know [18] that standard V-cycle multigrid does not have optimal convergence rate for the biharmonic problem. The good iteration numbers from Table 1 are a result of having a trivial geometry mapping. A block smoother is suggested in [18], where a block contains all the degrees of freedom (DoFs) associated with the boundary at one line. For the inner DoFs, normal point smoother is used. The number of iterations using this technique did not improve remarkably compared to Table 2.

We instead use block Gauss-Seidel on the whole domain, where the block size equal the number of DoFs belonging to one of the boundary sides. That is; the block size is increasing with the refinement. The results are displayed in Table 3. From the table, we see that we have h-independent iteration numbers. However, this block Gauss-Seidel smoother is computational expensive when the block sizes are large.

6.2 Condition numbers for different pairs of U_h and W_h

As mentioned in Section 5.3, we want to choose U_h and W_h such that $\tilde{A}_h$, see Equation (41), is equivalent to $A_h^T M_h^{-1} A_h$. In this subsection we present some condition numbers, κ, of $\tilde{A}_h \left(A_h^T M_h^{-1} A_h\right)^{-1}$, which will give an indication of the equivalency. Ω is the same quarter annulus, which is used for the biharmonic problem above.

We let $U_h = \tilde{S}^{p,p}_{p-1,p-1}$, for $p \geq 2$ and we choose W_h accordingly. If $\Delta U_h \subset W_h$; that is, we choose $W_h = S^{p,p}_{p-3,p-3}$, the condition number is 1. Which is not surprising since $\tilde{A}_h = A_h^T M_h^{-1} A_h$, for these discretization spaces. If we let $W_h = S^{p,p}_{p-1,p-1}$; that is,

[3]Since we are using B-splines and not NURBS, Ω is an approximation to the quarter annulus.

$h \setminus p$	2	3	4	5	6
2	1.19	1.02	1.01	1.01	1.01
3	1.27	1.05	1.03	1.02	1.02
4	1.31	1.06	1.03	1.02	1.02
5	1.32	1.07	1.04	1.03	1.02

Table 4: Condition number κ, when $U_h = \tilde{S}^{p,p}_{p-1,p-1}$ and $W_h = S^{p,p}_{p-1,p-1}$. Ω is a quarter annulus.

$h \setminus p$	2	3	4	5	6
2	1.22	1.53	1.87	2.18	2.48
3	1.26	1.48	1.87	2.29	2.73
4	1.27	1.47	1.85	2.26	2.71
5	1.28	1.47	1.84	2.27	2.71

Table 5: Condition number κ, when $U_h = \tilde{S}^{p,p}_{p-1,p-1}$ and $W_h = S^{p-2,p-2}_{p-3,p-3}$. Ω is a quarter annulus.

W_h is equal to U_h, we know from Section 5.2 that this gives a stable discretization. However, we have not shown that $\tilde{A}_h$ and $A_h^T M_h^{-1} A_h$ are equivalent. Table 4 shows the condition numbers when $U_h = \tilde{S}^{p,p}_{p-1,p-1}$ and $W_h = S^{p,p}_{p-1,p-1}$, for different polynomial order, p, and number of uniform h-refinements. From the table we see that the condition numbers is slightly higher than 1 and independent of p and h. This is strong indication that $\tilde{A}_h$ and $A_h^T M_h^{-1} A_h$ are in fact equivalent for our domain Ω. Table 5 shows the condition numbers when $U_h = \tilde{S}^{p,p}_{p-1,p-1}$ and $W_h = S^{p-2,p-2}_{p-3,p-3}$; that is, the polynomial order of W_h is two degrees lower than that of U_h. The table shows h-independence and small dependency on p. These condition numbers indicate that $\tilde{A}_h$ and $A_h^T M_h^{-1} A_h$ are equivalent. When decreasing the polynomial order of W_h further (compared too U_h), the condition numbers became very large, which indicates that $\tilde{A}_h$ and $A_h^T M_h^{-1} A_h$ are no longer equivalent.

6.3 The KKT system

We solve Problem 5.2 by setting $u = \sin(2\pi x)\sin(4\pi y)$ on the boundary and $d_n = \nabla u \cdot \mathbf{n}$, where $\mathbf{n}$ is the outward pointing normal. The domain is the same quarter annulus, which was used for the biharmonic problem. We let $U_h = \tilde{S}^{p,p}_{p-1,p-1}$ and $W_h = S^{p-2,p-2}_{p-3,p-3}$, for $p \geq 2$. Note that for the pair of spaces, U_h and W_h, we have not shown that they satisfy the conditions in Theorem 3.2 or proven the inf-sup condition, Equation (42). However, the condition numbers in Table 5 indicates that they are stable.

Table 6 and Table 7 show the iteration numbers using the canonical preconditioner for $\alpha A_h^T M_h^{-1} A_h + K_h$ and $\alpha \tilde{A}_h + K_h$, respectively; that is, we use the exact preconditioner. This is done to give a reference of how good our approximated preconditioner is. Each columns of the tables represent an α value and the rows are the number of h-refinements. The numbers in the last column are the sizes of the system (number of DoFs). Table 6 and Table 7 show that the replacement of $A_h^T M_h^{-1} A_h$ with the biharmonic $\tilde{A}_h$ works well for the high values of α. For low values of α, Table 6 shows a reduction of iteration numbers. The biharmonic replacement does not mimic this behavior too well. However, the iteration numbers are not growing when α becomes small, so the preconditioner is α-robust.

$r\backslash\alpha$	1	0.1	0.01	1e-3	1e-5	1e-7	DoFs
3	31	51	47	35	18	14	192
4	30	52	58	44	21	14	768
5	31	54	63	55	23	14	3072
6	31	55	64	59	27	15	12288

Table 6: $U_h = \tilde{S}^{2,2}_{1,1}$ and $W_h = S^{0,0}_{-1,-1}$. Using the canonical preconditioner of $A_h^T M_h^{-1} A_h$.

$r\backslash\alpha$	1	0.1	0.01	1e-3	1e-5	1e-7	DoFs
3	36	56	63	58	46	46	192
4	37	58	66	62	48	46	768
5	38	60	68	65	48	44	3072
6	40	60	70	65	52	45	12288

Table 7: $U_h = \tilde{S}^{2,2}_{1,1}$ and $W_h = S^{0,0}_{-1,-1}$. Using the canonical preconditioner of $\tilde{A}_h$.

$r\backslash\alpha$	1	0.1	0.01	1e-3	1e-5	1e-7	DoFs
3	37	56	64	60	51	51	192
4	39	58	68	64	54	54	768
5	39	60	70	66	55	54	3072
6	42	60	72	68	55	55	12288
7	42	62	73	71	58	53	49152

Table 8: $U_h = \tilde{S}^{2,2}_{1,1}$ and $W_h = S^{0,0}_{-1,-1}$. Using multigrid with block smoother.

$r\backslash\alpha$	1	0.1	0.01	1e-3	1e-5	1e-7	DoFs
3	67	80	84	75	63	67	300
4	72	80	89	79	71	75	972
5	74	83	90	81	69	73	3468
6	77	85	93	81	69	72	13068
7	77	87	96	89	69	70	50700

Table 9: $U_h = \tilde{S}^{4,4}_{3,3}$ and $W_h = S^{2,2}_{1,1}$. Using multigrid with block smoother.

$r\backslash\alpha$	1	0.1	0.01	1e-3	1e-5	1e-7	DoFs
3	164	180	180	166	147	157	432
4	181	190	188	162	139	153	1200
5	188	184	183	164	141	148	3888
6	198	196	192	170	144	142	13872
7	205	205	200	183	150	147	52272

Table 10: $U_h = \tilde{S}^{6,6}_{5,5}$ and $W_h = S^{4,4}_{3,3}$. Using multigrid with block smoother.

Table 8 shows the iteration numbers when using standard multigrid on the matrix $\alpha\tilde{A}_h + K_h$ with the block smoother described in Section 6.1. From Table 8, it seems that the iteration numbers are h-independent. The table has similar values to the canonical preconditioner with $\tilde{A}_h$, Table 7, with only a few iterations higher. Hence, the multigrid approximation of $\alpha\tilde{A}_h + K_h$ with the block smoother seems to work well.

Table 9 and 10 show the iteration numbers of the approximated preconditioner, for $p = 4$ and $p = 6$, respectively. Here we observe the same behavior as for the $p = 2$ case and also the growth of iteration numbers for higher p, which is expected, as mention in the biharmonic example, Table 3.

7 Discussion

In Theorem 3.2 we presented a set of conditions which ensures a stable discretization of Problem 2.2. This is done by utilizing the theory in [19]. The robust preconditioner and the norms which follows from Theorem 3.2 were originally shown in [13]. We believe that the conditions in Theorem 3.2 are easier to use to find stable discretization spaces, U_h and W_h. For the elliptic Problem 5.1, we used Theorem 3.2 to show that $U_h \subset \bar{H}^2(\Omega)$ and W_h equal to U_h (without the prescribed boundary condition) are stable discretization spaces. However, we have not able to prove that preconditioner component, $A_h^T M_h^{-1} A_h$, is equivalent to the discretized biharmonic operator, $\tilde{A}_h$. This is important, in order to be able to efficiently precondition the problem. The numerical result indicate that they are equivalent.

IGA is a natural choice for discretizing, since the discretization space for the state requires high smoothness and this is easily obtained with IGA, compared to other methods, like FEM. Another strength of IGA is the many geometries which can be exactly represented with B-splines and NURBS. We have only considered single patch geometries in this article. To use multipatch, one needs to ensure the continuity requirement also hold over the patch interface. In [12] a method for doing this is given.

The challenging component for creating an efficient preconditioner for the KKT system is $\alpha\tilde{A}_h + K_h$. We found that a standard geometric multigrid method for the biharmonic discretization, $\tilde{A}_h$, is not h-independent for non-trivial geometry mappings. The remedies given in [18], does not work for the IGA discretization. A block smoother was numerically tested and showed h-independence. However, the smoother is computationally expensive. To the authors knowledge, no efficient multigrid method has been developed yet for the biharmonic equation in the IGA setting, which is h and p-independent.

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