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A Robust Multigrid Method for Isogeometric Analysis using Boundary Correction

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Abstract

We consider geometric multigrid methods for the solution of linear systems arising from isogeometric discretizations of elliptic partial differential equations. For classical finite elements, such methods are well known to be fast solvers showing optimal convergence behavior. However, the naive application of multigrid to the isogeometric case results in significant deterioration of the convergence rates if the spline degree is increased.

Recently, a robust approximation error estimate and a corresponding inverse inequality for B-splines of maximum smoothness have been shown, both with constants independent of the spline degree. We use these results to construct multigrid solvers for discretizations based on B-splines with maximum smoothness which exhibit robust convergence rates.

Keywords: Isogeometric Analysis, geometric multigrid, robustness

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1. Introduction

Isogeometric Analysis (IgA), introduced by Hughes et al. [15], is an approach to the discretization of partial differential equations (PDEs) which aims to bring geometric modeling and numerical simulation closer together. The fundamental idea is to use spaces of B-splines or non-uniform rational B-splines (NURBS) both for the geometric description of the computational domain and as discretization spaces for the numerical solution of PDEs on such domains. As for classical finite element methods (FEM), this leads to linear systems with large, sparse matrices. A good approximation of the solution of the PDE requires sufficient refinement, which causes both the dimension and the condition number of the stiffness matrix to grow. At least for problems on three-dimensional

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domains or the space-time cylinder, the use of direct solvers does not seem feasible. The development of efficient linear solvers or preconditioners for such linear systems is therefore essential.

For classical FEM, it is well-known that hierarchical methods, like multigrid and multilevel methods, are very efficient and show optimal complexity, that is, the required number of iterations for reaching a fixed accuracy goal is independent of the grid size. In this case, the overall computational complexity of the method grows only linearly with the number of unknowns.

It therefore seems natural to extend these methods to IgA, and several results in this direction can be found in the literature. Multigrid methods for IgA based on classical concepts have been considered in [10, 13, 14], and a classical multilevel method in [2]. It has been shown early on that a standard approach to constructing geometric multigrid solvers for IgA leads to methods which are robust in the grid size [10]. However, it has been observed that the resulting convergence rates deteriorate significantly when the spline degree is increased. Even for moderate choices like splines of degree four, too many iterations are required for practical purposes.

Recently, in [12] some progress has been made by using a Richardson method preconditioned with the mass matrix as a smoother (*mass-Richardson smoother*). The idea is to carry over the concept of operator preconditioning to multigrid smoothing: here the (inverse of) the mass matrix can be understood as a Riesz isomorphism representing the standard L^2-norm, the Hilbert space where the classical multigrid convergence analysis is developed. Local Fourier analysis indicates that a multigrid method equipped with such a smoother should show convergence rates that are independent of both the grid size and the spline degree. However, numerical results indicate that the proposed method is not robust in the spline degree in practice. This is due to boundary effects, which cannot be captured by local Fourier analysis. Similar techniques have been used to construct symbol-based multigrid approaches for IgA, see [8, 7, 9].

In the present paper, we take a closer look at the origin of these boundary effects and introduce a boundary correction that deals with these effects. We prove that the multigrid method equipped with a properly corrected mass-Richardson smoother converges robustly both in the grid size and the spline degree for one- and two-dimensional problems. For the proof, we make use of the results of the recent paper [16], where robust approximation error estimates and robust inverse estimates for splines of maximum smoothness have been shown. We present numerical results that illustrate the theoretical results. Throughout the paper we restrict ourselves to the case of splines with maximum continuity, which is of particular interest in IgA.

The bulk of our analysis is first carried out in the one-dimensional setting. While multigrid solvers are typically not interesting in this setting from a practical point of view, the tensor product structure of the spline spaces commonly used in IgA lends itself very well to first analyzing the one-dimensional case and then extending the results into higher dimensions. In the present work, we extend the ideas from the one-dimensional to the two-dimensional case and obtain a robust and efficiently realizable smoother for that setting.

The paper is organized as follows. In Section 2, we introduce the spline spaces used for the discretization and the elliptic model problem. A general multigrid framework is introduced in Section 3. This framework requires a proper choice of the smoother, which is discussed in detail for one-dimensional domains in Section 4. In Section 5, we extend these results to the case of two-dimensional domains. Some details on the numerical realization of the proposed smoother as well as numerical experiments illustrating the theory are given in Section 6. In Section 7, we close with some concluding remarks. A few proofs are postponed to the Appendix.

2. Preliminaries

2.1. B-splines and tensor product B-splines

First, we consider the case of one-dimensional domains and assume, without loss of generality, $\Omega = (0,1)$. We introduce for any $\ell \in \mathbb{N}_0 = \{0,1,2,\ldots\}$ a uniform subdivision (grid) by splitting Ω into $n_\ell = n_0 2^\ell$ subintervals of length $h_\ell := \frac{1}{n_\ell} = \frac{1}{n_0} 2^{-\ell}$ by uniform dyadic refinement. On these grids we introduce spaces of spline functions with maximum smoothness as follows.

Definition 1. For $p \in \mathbb{N} := \{1,2,3,\ldots\}$, $S_{p,\ell}(0,1)$ is the space of all functions in $C^{p-1}(0,1)$ which are polynomials of degree p on each subinterval $((i-1)h_\ell, ih_\ell)$ for $i = 1,\ldots,n_\ell$. Here $C^m(\Omega)$ denotes the space of all continuous functions mapping $\Omega \to \mathbb{R}$ that are m times continuously differentiable.

We assume that the coarsest grid is chosen such that $n_0 > p$ holds. Note that the spaces are nested for fixed spline degree p, that is, $S_{p,\ell-1}(0,1) \subset S_{p,\ell}(0,1)$, and the number of degrees of freedom roughly doubles in each refinement step. The parameter ℓ will play the role of the grid level in the construction of our multigrid algorithm.

As a basis for $S_{p,\ell}(\Omega)$, we choose B-splines as described by, e.g., de Boor [6]. To this end, we introduce an open knot vector with the first and last knot repeated $p+1$ times each, $(0,\ldots,0,h_\ell,2h_\ell,\ldots,(n_\ell-1)h_\ell,1,\ldots,1)$, and define the normalized B-spline basis over this knot vector in the standard way. We denote the B-spline basis functions by $\{\varphi_{p,\ell}^{(1)},\ldots,\varphi_{p,\ell}^{(m_\ell)}\}$, where $m_\ell = \dim S_{p,\ell}(\Omega) = n_\ell + p$. They form a partition of unity, that is, $\sum_{j=1}^{m_\ell} \varphi_{p,\ell}^{(j)}(x) = 1$ for all $x \in \Omega$.

In higher dimensions, we will assume $\Omega = (0,1)^d$ with $d > 1$ and introduce tensor product B-spline basis functions of the form $(x,y) \mapsto \varphi_{p,\ell}^{(j_1)}(x)\varphi_{p,\ell}^{(j_2)}(y)$ for two dimensions and analogous functions for higher dimensions. The space spanned by these basis functions will again be denoted by $S_{p,\ell}(\Omega)$. The extension to the case where $\Omega \subset \mathbb{R}^d$ is the tensor product of d arbitrary bounded and open intervals is straightforward. Similarly, it is no problem to have different spline degrees, grid sizes, and/or number of subintervals for each dimension, as long as the grid sizes are approximately equal in all directions. However, for the sake of simplicity of the notation, we will always assume that the discretization is identical in each coordinate direction.

2.2. Model problem

For the sake of simplicity, we restrict ourselves to the following model problem. Let $\Omega = (0,1)^d$ and assume $f \in L^2(\Omega)$ to be a given function. Find a function $u : \Omega \to \mathbb{R}$ such that

$$-\Delta u + u = f \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

In variational form, this problem reads: find $u \in V := H^1(\Omega)$ such that

$$\underbrace{(\nabla u, \nabla v)_{L^2(\Omega)} + (u,v)_{L^2(\Omega)}}_{(u,v)_A :=} = \underbrace{(f,v)_{L^2(\Omega)}}_{\langle f,v\rangle :=} \qquad \forall v \in V.$$

Here and in what follows, $L^2(\Omega)$ is the standard Lebesgue space of square integrable functions and $H^1(\Omega)$ denotes the standard Sobolev space of weakly differentiable functions with derivatives in $L^2(\Omega)$.

Applying a Galerkin discretization using spline spaces, we obtain the following discrete problem: find $u_\ell \in V_\ell := S_{p,\ell}(\Omega) \subset V$ such that

$$(u_\ell, v_\ell)_A = \langle f, v_\ell \rangle \qquad \forall v_\ell \in V_\ell.$$

Using the B-spline basis introduced in the previous subsection, the discretized problem can be rewritten in matrix-vector notation,

$$A_\ell \underline{u}_\ell = \underline{f}_\ell, \tag{1}$$

where A_ℓ is the B-spline stiffness matrix. Here and in what follows, underlined symbols for primal variables, like $\underline{u}_\ell$ and $\underline{v}_\ell$, are the coefficient vectors representing the corresponding functions u_ℓ and v_ℓ with respect to the B-spline basis.

Remark 1. Our model problem may seem rather restrictive in the sense that we only allow for tensor product domains. In practical IgA problems, one often uses such domains as parameter domains and introduces a geometry mapping, usually given in the same basis as used for the discretization, which maps the parameter domain to the actual physical domain of interest. As long as the geometry mapping is well-behaved, a good solver on the parameter domain can be used as a preconditioner for the problem on the physical domain, and our model problem therefore captures all essential difficulties that arise in the construction of multigrid methods in this more general setting. Geometry mappings with singularities as well as multi-patch domains are beyond the scope of this paper.

3. A multigrid solver framework

3.1. Description of the multigrid algorithm

We now introduce a standard multigrid algorithm for solving the discretized equation (1) on grid level ℓ. Starting from an initial approximation $\underline{u}_\ell^{(0)}$, the next iterate $\underline{u}_\ell^{(1)}$ is obtained by the following two steps:

- *Smoothing procedure:* For some fixed number ν of smoothing steps, compute

$$\underline{u}_\ell^{(0,m)} := \underline{u}_\ell^{(0,m-1)} + \tau L_\ell^{-1} \left(\underline{f}_\ell - A_\ell \, \underline{u}_\ell^{(0,m-1)} \right) \qquad \text{for } m = 1, \ldots, \nu, \quad (2)$$

  where $\underline{u}_\ell^{(0,0)} := \underline{u}_\ell^{(0)}$. The choice of the matrix L_ℓ and the damping parameter $\tau > 0$ will be discussed below.

- *Coarse-grid correction:*

  - Compute the defect and restrict it to grid level $\ell-1$ using a restriction matrix $I_\ell^{\ell-1}$:
  $$\underline{r}_{\ell-1}^{(1)} := I_\ell^{\ell-1} \left(\underline{f}_\ell - A_\ell \, \underline{u}_\ell^{(0,\nu)} \right).$$
  - Compute the correction $\underline{p}_{\ell-1}^{(1)}$ by approximately solving the coarse-grid problem
  $$A_{\ell-1} \, \underline{p}_{\ell-1}^{(1)} = \underline{r}_{\ell-1}^{(1)}. \quad (3)$$
  - Prolongate $\underline{p}_{\ell-1}^{(1)}$ to the grid level ℓ using a prolongation matrix $I_{\ell-1}^\ell$ and add the result to the previous iterate:
  $$\underline{u}_\ell^{(1)} := \underline{u}_\ell^{(0,\nu)} + I_{\ell-1}^\ell \, \underline{p}_{\ell-1}^{(1)}.$$

As we have assumed nested spaces, the intergrid transfer matrices $I_{\ell-1}^\ell$ and $I_\ell^{\ell-1}$ can be chosen in a canonical way: $I_{\ell-1}^\ell$ is the canonical embedding and the restriction $I_\ell^{\ell-1}$ its transpose. In the IgA setting, the prolongation matrix $I_{\ell-1}^\ell$ can be computed by means of knot insertion algorithms.

If the problem (3) on the coarser grid is solved exactly (*two-grid method*), the coarse-grid correction is given by

$$\underline{u}_\ell^{(1)} := \underline{u}_\ell^{(0,\nu)} + I_{\ell-1}^\ell \, A_{\ell-1}^{-1} \, I_\ell^{\ell-1} \left(\underline{f}_\ell - A_\ell \, \underline{u}_\ell^{(0,\nu)} \right). \quad (4)$$

In practice, the problem (3) is approximately solved by recursively applying one step (*V-cycle*) or two steps (*W-cycle*) of the multigrid method. On the coarsest grid level ($\ell = 0$), the problem (3) is solved exactly by means of a direct method.

The crucial remaining task is the choice of the smoother. For multigrid methods for elliptic problems with a FEM discretization, it is common to use a damped Jacobi iteration or a Gauss-Seidel iteration as the smoother. This is also possible if an isogeometric discretization is used, but leads to significant deterioration in the convergence rates as p is increased ([10, 14]). This motivates to choose a non-standard smoother. In the sequel of this section, we derive conditions on the matrix L_ℓ, guaranteeing convergence, which we use in the following two sections to construct a smoother for the model problem.

3.2. Abstract multigrid convergence theory

As the two-grid method is a linear iteration scheme, its action on the error is fully described by the iteration matrix. Let $\underline{u}_\ell^*$ denote the exact solution of (1). Then the initial error $\underline{u}_\ell^* - \underline{u}_\ell^{(0)}$ and the error after one two-grid cycle $\underline{u}_\ell^* - \underline{u}_\ell^{(1)}$ are related by the equation

$$\underline{u}_\ell^* - \underline{u}_\ell^{(1)} = (I - T_{\ell-1}) S_\ell^\nu \, (\underline{u}_\ell^* - \underline{u}_\ell^{(0)}),$$

where

$$S_\ell = I - \tau L_\ell^{-1} A_\ell, \qquad T_{\ell-1} = I_{\ell-1}^{\ell} \, A_{\ell-1}^{-1} \, I_\ell^{\ell-1} A_\ell.$$

Observe that S_ℓ and $I - T_{\ell-1}$ are the iteration matrices of the smoother and of the coarse-grid correction, respectively.

Throughout the paper we use the following notations. For a symmetric and positive definite matrix Q_ℓ and a vector $\underline{u}_\ell$, we define $\|\underline{u}_\ell\|_{Q_\ell} := (Q_\ell \underline{u}_\ell, \underline{u}_\ell)^{1/2} = \|Q_\ell^{1/2} \underline{u}_\ell\|$, where $\|\cdot\|$ and $(\cdot,\cdot)$ are the Euclidean norm and scalar product, respectively. The associated matrix norms are denoted by the same symbols $\|\cdot\|_{Q_\ell}$ and $\|\cdot\|$. For ease of notation, we use the same symbols also for the norm of functions $u_\ell \in V_\ell$ and mean in this case their application to the coefficient vector, i.e., $\|u_\ell\|_{Q_\ell} := \|\underline{u}_\ell\|_{Q_\ell}$. As A_ℓ is the matrix representation of the bilinear form $(\cdot,\cdot)_A$, we have $\|u_\ell\|_{A_\ell} = \|u_\ell\|_A$, where $\|\cdot\|_A := (\cdot,\cdot)_A^{1/2}$ is the energy norm.

We assume that the matrix L_ℓ appearing in (2) is symmetric and positive definite. Following the classical line of, e.g., [1], we show convergence in that norm, i.e., we show that

$$q := \|(I - T_{\ell-1}) S_\ell^\nu\|_{L_\ell} < 1, \tag{5}$$

which obviously implies the q-linear convergence property

$$\|\underline{u}_\ell^* - \underline{u}_\ell^{(1)}\|_{L_\ell} \le q \|\underline{u}_\ell^* - \underline{u}_\ell^{(0)}\|_{L_\ell}.$$

To analyze (5), we use semi-multiplicity of norms and obtain

$$\begin{aligned} \|(I - T_{\ell-1}) S_\ell^\nu\|_{L_\ell} &= \|L_\ell^{1/2} (I - T_{\ell-1}) S_\ell^\nu L_\ell^{-1/2}\| \\ &\le \|L_\ell^{1/2} (I - T_{\ell-1}) A_\ell^{-1} L_\ell^{1/2}\| \, \|L_\ell^{-1/2} A_\ell S_\ell^\nu L_\ell^{-1/2}\|. \end{aligned} \tag{6}$$

Therefore, it suffices to prove the two conditions

$$\|L_\ell^{1/2} (I - T_{\ell-1}) A_\ell^{-1} L_\ell^{1/2}\| \le C_A, \qquad \text{(approximation property)} \tag{7}$$

$$\|L_\ell^{-1/2} A_\ell S_\ell^\nu L_\ell^{-1/2}\| \le C_S \nu^{-1}, \qquad \text{(smoothing property)} \tag{8}$$

cf. [11, eq. (6.1.5)] for this splitting. Then (5) follows immediately for $\nu > C_A C_S$, that is, if sufficiently many smoothing steps are applied. As we are interested in robust convergence, the constants C_A and C_S should be independent of both the grid size and the spline degree.

The approximation property (7) is a direct consequence of the following approximation error estimate. Observe that $T_{\ell-1}$ is the matrix representation of the A-orthogonal projector from V_ℓ to $V_{\ell-1}$. For simplicity, we use the same symbol $T_{\ell-1}$ also for the A-orthogonal projector from V_ℓ to $V_{\ell-1}$ itself and, later on, also for the A-orthogonal projector from the whole space $V = H^1(\Omega)$ to $V_{\ell-1}$.

Lemma 1. *The approximation error estimate*

$$\|(I - T_{\ell-1})u_\ell\|_{L_\ell}^2 \leq C_A \|u_\ell\|_A^2 \qquad \forall u_\ell \in V_\ell \tag{9}$$

is equivalent to the approximation property (7) *with the same constant* C_A.

Proof. The estimate (9) can be rewritten as

$$\|X_\ell\| \leq C_A^{1/2} \quad \text{with} \quad X_\ell := L_\ell^{1/2}(I - T_{\ell-1})A_\ell^{-1/2},$$

which is equivalent to $\|X_\ell X_\ell^T\| \leq C_A$. We have

$$X_\ell X_\ell^T = L_\ell^{1/2}(I - T_{\ell-1})A_\ell^{-1}(I - T_{\ell-1})^T L_\ell^{1/2} = L_\ell^{1/2}(I - T_{\ell-1})^2 A_\ell^{-1} L_\ell^{1/2},$$

where we used $((I - T_{\ell-1})A_\ell^{-1})^T = (I - T_{\ell-1})A_\ell^{-1}$, which follows from $I - T_{\ell-1}$ being self-adjoint with respect to A_ℓ. Since $I - T_{\ell-1}$ is a projector, the statement follows. □

The following lemma states that the smoothing property is a direct consequence of an inverse inequality.

Lemma 2. *Let A_ℓ and L_ℓ be symmetric and positive definite matrices. Assume that the inequality*

$$\|u_\ell\|_A^2 \leq C_I \|u_\ell\|_{L_\ell}^2 \qquad \forall u_\ell \in V_\ell \tag{10}$$

holds. Then for $\tau \in (0, C_I^{-1}]$, the smoother (2) *satisfies the smoothing property* (8) *with $C_S = \tau^{-1}$. Under the same assumptions, $\|S_\ell\|_{A_\ell} \leq 1$ holds.*

Proof. The proof is based on [11, Lemma 6.2.1]. From (10), we immediately obtain $\tau^{-1} \geq \|A_\ell^{1/2} L_\ell^{-1/2}\|^2 = \rho(L_\ell^{-1/2} A_\ell L_\ell^{-1/2})$, where $\rho(\cdot)$ is the spectral radius. Observe that

$$L_\ell^{-1/2} A_\ell S_\ell^\nu L_\ell^{-1/2} = L_\ell^{-1/2} A_\ell (I - \tau L_\ell^{-1} A_\ell)^\nu L_\ell^{-1/2} = \bar{A}_\ell (I - \tau \bar{A}_\ell)^\nu$$

with $\bar{A}_\ell := L_\ell^{-1/2} A_\ell L_\ell^{-1/2}$ and that $\bar{A}_\ell(I - \tau\bar{A}_\ell)^\nu$ is symmetric. Furthermore, we have the spectral radius bound $\rho(\tau\bar{A}_\ell) = \tau\rho(L_\ell^{-1/2} A_\ell L_\ell^{-1/2}) \leq 1$. Thus,

$$\begin{aligned}
&\|L_\ell^{-1/2} A_\ell S_\ell^\nu L_\ell^{-1/2}\| \\
&\quad = \rho(\bar{A}_\ell(I - \tau\bar{A}_\ell)^\nu) = \sup_{\lambda \in \sigma(\bar{A}_\ell)} \lambda(1 - \tau\lambda)^\nu \leq \tau^{-1} \sup_{\mu \in [0,1]} \mu(1 - \mu)^\nu \\
&\quad = \tau^{-1} \left(\frac{\nu}{1 + \nu}\right)^\nu \frac{1}{1 + \nu} \leq \frac{\tau^{-1}}{\nu},
\end{aligned}$$

which shows (8). Here $\sigma(\cdot)$ denotes the spectrum of a matrix. Similarly we have $\|S_\ell\|_{A_\ell} = \rho(I - \tau\bar{A}_\ell) \leq 1$. □

Observe that estimate (10) is, indeed, of the typical form of an inverse inequality, if L_ℓ is a (properly scaled) mass matrix representing the L^2 inner product.

In view of Lemmas 1 and 2, we can state sufficient conditions for the convergence of a two-grid method.

Theorem 3. *Assume that there are constants C_A and C_I, independent of the grid size and the spline degree, such that the approximation error estimate* (9) *and the inverse inequality* (10) *hold. Then the two-grid method converges for the choice $\tau \in (0, C_I^{-1}]$ and $\nu > \nu_0 := \tau^{-1} C_A$ with rate $q = \nu_0/\nu$.*

Proof. The statement on the convergence of the two-grid method follows directly from (6), Lemma 1, and Lemma 2. □

The assumptions of Theorem 3 are also sufficient to prove convergence of a symmetrical variant of the W-cycle multigrid method. Here, in addition to the smoothing steps before the coarse-grid correction (*pre-smoothing*), we perform an equal number of smoothing steps after the coarse-grid correction (*post-smoothing*).

Theorem 4. *Under the assumptions of Theorem 3, the W-cycle multigrid method with $\nu/2$ pre- and $\nu/2$ post-smoothing steps converges in the energy norm $\|\cdot\|_A$ for the choice $\tau \in (0, C_I^{-1}]$ and $\nu > 4\nu_0 = 4\tau^{-1}C_A$ with rate $q = 2\nu_0/\nu$.*

Proof. Theorem 3 states that

$$\|(I - T_{\ell-1})S_\ell^\nu\|_{L_\ell} \leq \frac{\tau^{-1}C_A}{\nu}.$$

As $S_\ell^{\nu/2}(I - T_{\ell-1})S_\ell^{\nu/2}$ is self-adjoint in the scalar product $(\cdot,\cdot)_{A_\ell}$, this implies

$$\|S_\ell^{\nu/2}(I - T_{\ell-1})S_\ell^{\nu/2}\|_{A_\ell} = \rho((I - T_{\ell-1})S_\ell^\nu) \leq \|(I - T_{\ell-1})S_\ell^\nu\|_{L_\ell} \leq \frac{\tau^{-1}C_A}{\nu}. \quad (11)$$

The iteration matrix of the W-cycle multigrid method is recursively given by

$$W_\ell = S_\ell^{\nu/2}(I - I_{\ell-1}^\ell(I - W_{\ell-1}^2)A_{\ell-1}^{-1}I_\ell^{\ell-1}A_\ell)S_\ell^{\nu/2} \qquad \text{for } \ell > 0,$$

and $W_0 = 0$. Using the triangle inequality and semi-multiplicativity of norms, we obtain for the convergence rate

$$\begin{aligned} q_\ell = \|W_\ell\|_{A_\ell} &= \|S_\ell^{\nu/2}(I - I_{\ell-1}^\ell(I - W_{\ell-1}^2)A_{\ell-1}^{-1}I_\ell^{\ell-1}A_\ell)S_\ell^{\nu/2}\|_{A_\ell} \\ &\leq \|S_\ell^{\nu/2}(I - T_{\ell-1})S_\ell^{\nu/2}\|_{A_\ell} + \|S_\ell^{\nu/2}I_{\ell-1}^\ell W_{\ell-1}^2 A_{\ell-1}^{-1} I_\ell^{\ell-1} A_\ell S_\ell^{\nu/2}\|_{A_\ell} \\ &\leq \|S_\ell^{\nu/2}(I - T_{\ell-1})S_\ell^{\nu/2}\|_{A_\ell} + \|S_\ell\|_{A_\ell}^\nu \|A_\ell^{1/2} I_{\ell-1}^\ell A_{\ell-1}^{-1/2}\|^2 \|W_{\ell-1}\|_{A_{\ell-1}}^2. \end{aligned}$$

As the spaces V_ℓ are nested, we obtain $A_{\ell-1} = I_\ell^{\ell-1}A_\ell I_{\ell-1}^\ell$. Thus it follows $\|A_\ell^{1/2}I_{\ell-1}^\ell A_{\ell-1}^{-1/2}\|^2 = \rho(I_\ell^{\ell-1}A_\ell I_{\ell-1}^\ell A_{\ell-1}^{-1}) = 1$. Lemma 2 states that $\|S_\ell\|_{A_\ell} \leq 1$. Using these two statements, (11) and $q_{\ell-1} = \|W_{\ell-1}\|_{A_{\ell-1}}$, we obtain

$$q_\ell \leq \frac{\nu_0}{\nu} + q_{\ell-1}^2.$$

Using the assumption $\nu \geq 4\nu_0$, one shows by induction that $q_\ell \leq 2\nu_0/\nu$. □

4. A robust multigrid method for one-dimensional domains

4.1. Robust estimates for a subspace of the spline space

For the model problem, we have $V = H^1(\Omega)$ and

$$\| \cdot \|_A = \| \cdot \|_{H^1(\Omega)}. \tag{12}$$

The standard multigrid convergence analysis as introduced by Hackbusch [11] gives convergence in a (properly scaled) L^2-norm,

$$\| \cdot \|_{L_\ell} = h_\ell^{-1} \| \cdot \|_{L^2(\Omega)}, \tag{13}$$

and thus the choice $L_\ell = h_\ell^{-2} M_\ell$, where M_ℓ is the mass matrix, consisting of pairwise L^2-scalar products of the basis functions.

To show robust convergence of the multigrid solver with a Jacobi smoother, one would show that the diagonal of A_ℓ and $h_\ell^{-2} M_\ell$ are spectrally equivalent. This equivalence holds robustly in the grid size, however the involved constants deteriorate with increasing spline degree p. This issue is closely related to the so-called condition number of the B-spline basis (see, e.g., [4]). The growth of the condition number with p explains why standard Jacobi iteration and Gauss-Seidel iteration do not work well for B-splines.

The aforementioned equivalence is not necessary for a direct application of the choice $L_\ell = h_\ell^{-2} M_\ell$, which is the mass-Richardson smoother already studied in [12]. Local Fourier analysis indicates that this smoother should lead to robust convergence of the multigrid solver. However, the numerical experiments in [12] show that this is not the case and iteration numbers still deteriorate with p. The reason for this effect is motivated as follows.

With the choice (12) and (13), the condition (10) reads

$$\|u_\ell\|_{H^1(\Omega)} \le C_I^{1/2} h_\ell^{-1} \|u_\ell\|_{L^2(\Omega)} \qquad \forall u_\ell \in V_\ell = S_{p,\ell}(\Omega). \tag{14}$$

In other words, it is required that the spline space satisfies an inverse inequality with a constant that is independent of the grid size and the spline degree. However, such a robust inverse inequality does not hold, as the counterexample given in [16] shows: for each $p \in \mathbb{N}$ and each $\ell \in \mathbb{N}_0$, there exists a spline $w_\ell \in S_{p,\ell}(\Omega)$ with

$$|w_\ell|_{H^1(\Omega)} \ge p h_\ell^{-1} \|w_\ell\|_{L^2(\Omega)}. \tag{15}$$

As we have to choose $\tau \le C_I^{-1}$ for the smoother not to diverge, the existence of such splines w_ℓ implies that one has to choose $\tau = \mathcal{O}(p^{-2})$, which causes the convergence rates to deteriorate. The splines w_ℓ in the counterexample are non-periodic and therefore cannot be captured by local Fourier analysis and similar tools which consider only the periodic setting.

In [16] it was shown that a robust inverse estimate of the form (14) does hold for the following large subspace of $S_{p,\ell}(\Omega)$.

Definition 2. Let $\Omega = (0,1)$. We denote by $\widetilde{S}_{p,\ell}(\Omega)$ the space of all $u_\ell \in S_{p,\ell}(\Omega)$ whose odd derivatives of order less than p vanish at the boundary,

$$\frac{\partial^{2l+1}}{\partial x^{2l+1}} u_\ell(0) = \frac{\partial^{2l+1}}{\partial x^{2l+1}} u_\ell(1) = 0 \quad \text{ for all } l \in \mathbb{N}_0 \text{ with } 2l+1 < p.$$

The space $\widetilde{S}_{p,\ell}(\Omega)$ is almost as large as $S_{p,\ell}(\Omega)$: their dimensions differ by p (for p even) or $p-1$ (for p odd).

Theorem 5 ([16]). *For all spline degrees $p \in \mathbb{N}$ and all grid levels $\ell \in \mathbb{N}_0$, we have the inverse inequality*

$$|u_\ell|_{H^1(\Omega)} \leq 2\sqrt{3} h_\ell^{-1} \|u_\ell\|_{L^2(\Omega)} \qquad \forall u_\ell \in \widetilde{S}_{p,\ell}(\Omega).$$

This result makes it clear that the counterexample (15) describes the effect of only a few outliers (see also [3] on the topic of spectral outliers in IgA), connected to the boundary. We make use of this fact by constructing a mass-Richardson smoother with a (low-rank) boundary correction.

Using $\|u_\ell\|_A^2 = |u_\ell|_{H^1(\Omega)}^2 + \|u_\ell\|_{L^2(\Omega)}^2$ and $h_\ell \leq 1$, the inverse inequality implies

$$\|u_\ell\|_A \leq c \|u_\ell\|_{h_\ell^{-2} M_\ell} \qquad \forall u_\ell \in \widetilde{S}_{p,\ell}(\Omega). \tag{16}$$

Here and in the sequel, we use c to refer to a generic constant which is independent of both the grid level ℓ and the spline degree p.

In addition to the inverse estimate, also an approximation property for the subspace $\widetilde{S}_{p,\ell}(\Omega)$ was proved in [16]. We slightly refine the result here as follows.

Theorem 6. *We have the approximation error estimate*

$$\|(I - \widetilde{T}_\ell) u\|_{L^2(\Omega)} \leq 2\sqrt{2} h_\ell \|u\|_{H^1(\Omega)} \qquad \forall u \in H^1(\Omega),$$

where $\widetilde{T}_\ell : H^1(\Omega) \to \widetilde{S}_{p,\ell}(\Omega)$ is the A-orthogonal projector.

The proof for this theorem is given in the Appendix.

Using (16) and Theorem 6, we immediately obtain assumptions (10) and (9) of Theorem 3 with the choice $V_\ell = \widetilde{S}_{p,\ell}(\Omega)$ and $L_\ell = h_\ell^{-2} M_\ell$. Thus, a two-grid method using the mass smoother would be robust in this subspace. However, the problems we are interested in are typically discretized in the full spline space $S_{p,\ell}(\Omega)$. We extend the choice of L_ℓ robustly to such problems in the following.

4.2. Extension of the inverse inequality to the entire spline space

In this section, we modify the mass matrix in such a way as to satisfy a robust inverse inequality in the entire spline space. This is done by means of an additional term which is essentially a discrete harmonic extension. We set

$$\widetilde{L}_\ell := h_\ell^{-2} M_\ell + (I - \widetilde{T}_\ell)^T A (I - \widetilde{T}_\ell).$$

Here, the second term corrects for the violation of the inverse inequality in the A-orthogonal complement of $\widetilde{S}_{p,\ell}(\Omega)$ by incorporating the original operator A. This term satisfies the energy minimization property

$$\|u_\ell\|_{(I-\widetilde{T}_\ell)^T A (I-\widetilde{T}_\ell)} = \|(I-\widetilde{T}_\ell)u_\ell\|_A = \inf_{v_\ell \in \widetilde{S}_{p,\ell}(\Omega)} \|u_\ell + v_\ell\|_A.$$

We now prove the inverse inequality for the modified norm $\|\cdot\|_{\widetilde{L}_\ell}$. The proof requires both the inverse inequality and the approximation error estimate in $\widetilde{S}_{p,\ell}(\Omega)$.

Lemma 7. *We have the inverse inequality*

$$\|u_\ell\|_A \leq c\|u_\ell\|_{\widetilde{L}_\ell} \qquad \forall u_\ell \in S_{p,\ell}(\Omega)$$

with a constant c which is independent of ℓ and p.

Proof. Let $u_\ell \in S_{p,\ell}(\Omega)$. Using the triangle inequality, the inverse inequality (16) and again the triangle inequality we obtain

$$\begin{aligned}\|u_\ell\|_A &\leq \|\widetilde{T}_\ell u_\ell\|_A + \|(I-\widetilde{T}_\ell)u_\ell\|_A \leq c h_\ell^{-1}\|\widetilde{T}_\ell u_\ell\|_{M_\ell} + \|(I-\widetilde{T}_\ell)u_\ell\|_A \\ &\leq c h_\ell^{-1}\|u_\ell\|_{M_\ell} + c h_\ell^{-1}\|(I-\widetilde{T}_\ell)u_\ell\|_{M_\ell} + \|(I-\widetilde{T}_\ell)u_\ell\|_A.\end{aligned}$$

By Theorem 6, we have $\|(I-\widetilde{T}_\ell)u_\ell\|_{M_\ell} = \|(I-\widetilde{T}_\ell)^2 u_\ell\|_{M_\ell} \leq c h_\ell \|(I-\widetilde{T}_\ell)u_\ell\|_A$, and it follows

$$\|u_\ell\|_A \leq c\left(h_\ell^{-1}\|u_\ell\|_{M_\ell} + \|(I-\widetilde{T}_\ell)u_\ell\|_A\right).$$

The statement follows since $\|(I-\widetilde{T}_\ell)u_\ell\|_A = \|u_\ell\|_{(I-\widetilde{T}_\ell)^T A_\ell (I-\widetilde{T}_\ell)}$. □

4.3. A robust multigrid method for the whole spline space

We first show that Theorem 6 can be easily extended to the projection into the entire spline space.

Lemma 8. *The approximation error estimate*

$$\|(I-T_\ell)u\|_{L^2(\Omega)} \leq c h_\ell \|u\|_{H^1(\Omega)} \qquad \forall u \in H^1(\Omega),$$

holds, where $T_\ell : H^1(\Omega) \to S_{p,\ell}(\Omega)$ is the A-orthogonal projector.

Proof. As $\widetilde{S}_{p,\ell}(\Omega) \subset S_{p,\ell}(\Omega)$, we have $\widetilde{T}_\ell = \widetilde{T}_\ell T_\ell$. This identity, the triangle inequality, Theorem 6, and the stability of the A-orthogonal projector T_ℓ yield

$$\begin{aligned}\|(I-T_\ell)u\|^2_{L^2(\Omega)} &\leq 2(\|(I-\widetilde{T}_\ell)u\|^2_{L^2(\Omega)} + \|(I-\widetilde{T}_\ell)T_\ell u\|^2_{L^2(\Omega)}) \\ &\leq 2c h_\ell(\|u\|^2_{H^1(\Omega)} + \|T_\ell u\|^2_{H^1(\Omega)}) \leq 4c h_\ell \|u\|^2_{H^1(\Omega)} \quad \forall u \in H^1(\Omega).\end{aligned}$$ □

The robust inverse inequality (Lemma 7) together with the approximation property (Lemma 8) allow us to prove robust convergence of the following multigrid method for the space $V_\ell := S_{p,\ell}(\Omega)$. It turns out that we also obtain robust convergence if we do not project into $\widetilde{S}_{p,\ell}(\Omega)$, but into a subspace $S^I_{p,\ell}(\Omega)$, which may be easier to handle in practice.

Theorem 9. *Let $S^I_{p,\ell}(\Omega) \subseteq \widetilde{S}_{p,\ell}(\Omega)$ a subspace and $T^I_\ell : S_{p,\ell}(\Omega) \to S^I_{p,\ell}(\Omega)$ the A-orthogonal projector. Consider the smoother* (2) *with the choice*

$$L_\ell := h_\ell^{-2} M_\ell + (I - T^I_\ell)^T A_\ell (I - T^I_\ell). \tag{17}$$

Then there exists a damping parameter $\tau > 0$ and a choice of $\nu_0 > 0$, both independent of ℓ and p, such that the two-grid method with $\nu > \nu_0$ smoothing steps converges with rate $q = \nu_0/\nu$.

With the same choice of τ and with $\nu > 4\nu_0$, also the W-cycle multigrid method with $\nu/2$ pre- and $\nu/2$ post-smoothing steps converges in the energy norm with convergence rate $q = 2\nu_0/\nu$.

Proof. We show the assumptions of Theorem 3. The statements then follow from Theorems 3 and 4.

Proof of assumption (10)**.** Let $u_\ell \in S_{p,\ell}(\Omega)$. Lemma 7 states

$$\|u_\ell\|_A^2 \le c(\|u_\ell\|^2_{h_\ell^{-2} M_\ell} + \|(I - \widetilde{T}_\ell)u_\ell\|_A^2).$$

Using the energy-minimizing property of the projection operators, we obtain

$$\begin{aligned}\|(I - \widetilde{T}_\ell)u_\ell\|_A &= \inf_{v_\ell \in \widetilde{S}_{p,\ell}(\Omega)} \|u_\ell - v_\ell\|_A \\ &\le \inf_{v_\ell \in S^I_{p,\ell}(\Omega)} \|u_\ell - v_\ell\|_A = \|(I - T^I_\ell)u_\ell\|_A\end{aligned}$$

since $S^I_{p,\ell}(\Omega) \subseteq \widetilde{S}_{p,\ell}(\Omega)$. Combining these estimates, we get the desired statement

$$\|u_\ell\|_A^2 \le c\|u_\ell\|^2_{L_\ell}. \tag{18}$$

Proof of assumption (9)**.** Let $u_\ell \in S_{p,\ell}(\Omega)$. Lemma 8 implies

$$\|(I - T_{\ell-1})u_\ell\|^2_{h_\ell^{-2} M_\ell} \le c\|u_\ell\|_A^2$$

Since both $I - T^I_\ell$ and $I - T_{\ell-1}$ are A-orthogonal projectors and thus are stable in the A-norm, we obtain the approximation error estimate (9) via

$$\begin{aligned}\|(I - T_{\ell-1})u_\ell\|^2_{L_\ell} &= \|(I - T_{\ell-1})u_\ell\|^2_{h_\ell^{-2} M_\ell} + \|(I - T^I_\ell)(I - T_{\ell-1})u_\ell\|_A^2 \\ &\le (c+1)\|u_\ell\|_A^2.\end{aligned}$$

□

The choice $S^I_{p,\ell}(\Omega) = \widetilde{S}_{p,\ell}(\Omega)$ in the above theorem is admissible and results in a robust multigrid method. However, the space $\widetilde{S}_{p,\ell}(\Omega)$ is somewhat difficult to work with in practice since it is not easy to represent in terms of the standard B-spline basis. Therefore, we choose a slightly smaller space for $S^I_{p,\ell}(\Omega)$ which is based on a simple splitting of the degrees of freedom. To this end, we split the spline space $S_{p,\ell}(\Omega)$ into "boundary" and "inner" functions,

$$S_{p,\ell}(\Omega) = S^\Gamma_{p,\ell}(\Omega) + S^I_{p,\ell}(\Omega),$$

where $S^{\Gamma}_{p,\ell}(\Omega)$ is spanned by the first p B-spline basis functions $\varphi^{(1)}_{p,\ell}, \ldots, \varphi^{(p)}_{p,\ell}$ and the last p basis functions $\varphi^{(m_\ell-p+1)}_{p,\ell}, \ldots, \varphi^{(m_\ell)}_{p,\ell}$, whereas the space $S^{I}_{p,\ell}(\Omega)$ is spanned by all the remaining basis functions. By construction, $S^{I}_{p,\ell}(\Omega)$ consists of all splines that vanish on the boundary together with all derivatives up to order $p-1$. Therefore, we have $S^{I}_{p,\ell}(\Omega) \subseteq \widetilde{S}_{p,\ell}(\Omega)$.

Due to this subspace relation, the inverse inequality (16) remains valid in $S^{I}_{p,\ell}(\Omega)$. However, the analogue of the approximation property Theorem 6 does not in general hold in this smaller space. The existence of the larger subspace $\widetilde{S}_{p,\ell}(\Omega)$ in which both properties hold is crucial for the proof of assumption (10) above, even though we do not directly use the space $\widetilde{S}_{p,\ell}(\Omega)$ in practice.

Concerning the practical realization of such a smoother, observe that we can reorder the degrees of freedom based on the splitting $S_{p,\ell}(\Omega) = S^{\Gamma}_{p,\ell}(\Omega) + S^{I}_{p,\ell}(\Omega)$ and write the matrix A_ℓ and the vector $\underline{u}_\ell$ in block structure as

$$A_\ell = \begin{pmatrix} A_{\Gamma\Gamma,\ell} & A^{T}_{I\Gamma,\ell} \\ A_{I\Gamma,\ell} & A_{II,\ell} \end{pmatrix}, \qquad \underline{u}_\ell = \begin{pmatrix} \underline{u}_{\Gamma,\ell} \\ \underline{u}_{I,\ell} \end{pmatrix}. \tag{19}$$

Using a matrix representation of the projector T^{I}_{ℓ}, direct computation yields

$$(I - T^{I}_{\ell})\underline{u}_\ell = \begin{pmatrix} \underline{u}_{\Gamma,\ell} \\ -A^{-1}_{II,\ell} A_{I\Gamma,\ell} \underline{u}_{\Gamma,\ell} \end{pmatrix},$$

and therefore the matrix L_ℓ can be represented using the Schur complement as

$$L_\ell = h_\ell^{-2} M_\ell + C_\ell := h_\ell^{-2} M_\ell + \begin{pmatrix} A_{\Gamma\Gamma,\ell} - A^{T}_{I\Gamma,\ell} A^{-1}_{II,\ell} A_{I\Gamma,\ell} & 0 \\ 0 & 0 \end{pmatrix}. \tag{20}$$

We remark that $\underline{u}_\ell^T C_\ell \underline{u}_\ell$ represents the (squared) energy of the discrete harmonic extension of $\underline{u}_{\Gamma,\ell}$ from $S^{\Gamma}_{p,\ell}(\Omega)$ to $S_{p,\ell}(\Omega)$.

5. A robust multigrid method for two-dimensional domains

In this section, we extend the theory presented in the previous section to the (more relevant) case of two-dimensional domains. As outlined in Remark 1, we restrict ourselves to problems without geometry mapping, i.e., problems on the unit square only. However, these approaches can be used as preconditioners for problems on general geometries if there is a regular geometry mapping.

In the following, we are interested in solving the problem (1) with $d = 2$. Here, we assume to have a tensor product space. Without loss of generality, we assume for sake of simplicity that $\Omega = (0,1)^2$ and

$$\mathcal{S}_{p,\ell}(\Omega) := S_{p,\ell}(0,1) \otimes S_{p,\ell}(0,1).$$

Below, we will use calligraphic letters to refer to matrices for the two-dimensional domain, whereas standard letters refer to matrices for the one-dimensional domain. Using the tensor product structure of the problem and the discretization, the mass matrix has the tensor product structure

$$\mathcal{M}_\ell = M_\ell \otimes M_\ell,$$

where $\otimes$ denotes the Kronecker product. To keep the notation simple, we assume that we have the same basis and thus the same matrix M_ℓ for both directions of the two-dimensional domain. However, this is not needed for the analysis.

The stiffness matrix in two dimensions has the representation

$$\mathcal{A}_\ell = K_\ell \otimes M_\ell + M_\ell \otimes K_\ell + M_\ell \otimes M_\ell, \tag{21}$$

where K_ℓ represents the scalar product $(\nabla u_\ell, \nabla v_\ell)_{L^2(\Omega)}$ in one dimension. This decomposition reflects that

$$(u_\ell, v_\ell)_A = (\partial_x u_\ell, \partial_x v_\ell)_{L^2(\Omega)} + (\partial_y u_\ell, \partial_y v_\ell)_{L^2(\Omega)} + (u_\ell, v_\ell)_{L^2(\Omega)}.$$

The problem of interest is now

$$\mathcal{A}_\ell \underline{u}_\ell = \underline{f}_\ell.$$

Here, the multigrid framework from Section 3 applies unchanged, only replacing standard letters by calligraphic letters. Again, we have to choose a suitable smoother by determining the matrix $\mathcal{L}_\ell$. To this end, we again have a look at the approximation error estimate and the inverse inequality.

Let $\widetilde{\mathcal{S}}_{p,\ell}(\Omega) := \widetilde{S}_{p,\ell}(0,1) \otimes \widetilde{S}_{p,\ell}(0,1)$ and let $\|\cdot\|_{\mathcal{A}}$ be the energy norm for the problem in two dimensions, i.e., such that $\|u_\ell\|_{\mathcal{A}} = \|\underline{u}_\ell\|_{\mathcal{A}_\ell}$ holds for all $u_\ell \in \mathcal{S}_{p,\ell}(\Omega)$. An approximation error estimate for the space $\widetilde{\mathcal{S}}_{p,\ell}(\Omega)$ was shown in [16, Theorem 8]. However, there, not the $\mathcal{A}$-orthogonal projector was analyzed. The following statement is analogous to Theorem 6 and can be proved based on the one-dimensional result.

Theorem 10. *The approximation error estimate*

$$\|(I - \widetilde{\mathcal{T}}_\ell) u_\ell\|_{L^2(\Omega)} \le c h_\ell \|u_\ell\|_{\mathcal{A}} \qquad \forall u_\ell \in \mathcal{S}_{p,\ell}(\Omega),$$

where $\widetilde{\mathcal{T}}_\ell : \mathcal{S}_{p,\ell}(\Omega) \to \widetilde{\mathcal{S}}_{p,\ell}(\Omega)$ is the $\mathcal{A}$-orthogonal projector, holds with a constant c which is independent of ℓ and p.

Proof. Theorem 6 implies that

$$\|(I - \widetilde{T}_\ell) u_\ell\|_{M_\ell} \le c h_\ell \|u_\ell\|_{A_\ell} \qquad \forall u_\ell \in S_{p,\ell}(0,1),$$

where $\widetilde{T}_\ell : S_{p,\ell}(0,1) \to \widetilde{S}_{p,\ell}(0,1)$ is the A-orthogonal projector. Stability of the A-orthogonal projector means that

$$\|(I - \widetilde{T}_\ell) u_\ell\|_{A_\ell} \le \|u_\ell\|_{A_\ell} \qquad \forall u_\ell \in S_{p,\ell}(0,1).$$

Combining these statements, we obtain

$$\|(I - \widetilde{T}_\ell \otimes \widetilde{T}_\ell) u_\ell\|_{A_\ell \otimes M_\ell + M_\ell \otimes A_\ell} \le c h_\ell \|u_\ell\|_{A_\ell \otimes A_\ell} \qquad \forall u_\ell \in \mathcal{S}_{p,\ell}(\Omega).$$

Note that $A_\ell = K_\ell + M_\ell$. Therefore, $\mathcal{A}_\ell = K_\ell \otimes M_\ell + M_\ell \otimes K_\ell + M_\ell \otimes M_\ell \leq A_\ell \otimes M_\ell + M_\ell \otimes A_\ell$ holds, where $\leq$ is understood in the spectral sense. Moreover,

$$\begin{aligned}\mathcal{A}_\ell \mathcal{M}_\ell^{-1} \mathcal{A}_\ell &= K_\ell M_\ell^{-1} K_\ell \otimes M_\ell + K_\ell \otimes K_\ell + K_\ell \otimes M_\ell + K_\ell \otimes K_\ell \\ &\quad + M_\ell \otimes K_\ell M_\ell^{-1} K_\ell + M_\ell \otimes K_\ell + K_\ell \otimes M_\ell + M_\ell \otimes K_\ell + M_\ell \otimes M_\ell \\ &\geq K_\ell \otimes K_\ell + K_\ell \otimes M_\ell + M_\ell \otimes K_\ell + M_\ell \otimes M_\ell = A_\ell \otimes A_\ell\end{aligned}$$

holds. Using these two estimates, we obtain further

$$\|(I - \widetilde{T}_\ell \otimes \widetilde{T}_\ell) u_\ell\|_{\mathcal{A}_\ell} \leq c h_\ell \|u_\ell\|_{\mathcal{A}_\ell \mathcal{M}_\ell^{-1} \mathcal{A}_\ell} \qquad \forall u_\ell \in \mathcal{S}_{p,\ell}(\Omega).$$

As the $\mathcal{A}$-orthogonal projection minimizes the norm $\|\cdot\|_{\mathcal{A}} = \|\cdot\|_{\mathcal{A}_\ell}$, we obtain

$$\|(I - \widetilde{\mathcal{T}}_\ell) u_\ell\|_{\mathcal{A}_\ell} \leq c h_\ell \|u_\ell\|_{\mathcal{A}_\ell \mathcal{M}_\ell^{-1} \mathcal{A}_\ell} \qquad \forall u_\ell \in \mathcal{S}_{p,\ell}(\Omega),$$

or in matrix form

$$\|\mathcal{A}_\ell^{1/2} (I - \widetilde{\mathcal{T}}_\ell) \mathcal{A}_\ell^{-1} \mathcal{M}_\ell^{1/2}\| \leq c h_\ell.$$

By transposing and using $\mathcal{A}_\ell^{-1} \widetilde{\mathcal{T}}_\ell^T = \widetilde{\mathcal{T}}_\ell \mathcal{A}_\ell^{-1}$, it follows

$$\|\mathcal{M}_\ell^{1/2} (I - \widetilde{\mathcal{T}}_\ell) \mathcal{A}_\ell^{-1/2}\| \leq c h_\ell$$

which can be equivalently rewritten as the desired result

$$\|(I - \widetilde{\mathcal{T}}_\ell) u_\ell\|_{\mathcal{M}_\ell} \leq c h_\ell \|u_\ell\|_{\mathcal{A}_\ell} \qquad \forall u_\ell \in \mathcal{S}_{p,\ell}(\Omega). \qquad \square$$

Completely analogously to the proof of Lemma 8, we can extend this approximation error estimate to

$$\|(I - \mathcal{T}_{\ell-1}) u_\ell\|_{L^2(\Omega)} \leq c h_\ell \|u_\ell\|_{\mathcal{A}} \qquad \forall u_\ell \in \mathcal{S}_{p,\ell}(\Omega), \tag{22}$$

where $\mathcal{T}_\ell : H^1(\Omega) \to \mathcal{S}_{p,\ell}(\Omega)$ is the $\mathcal{A}$-orthogonal projector.

From [16, Theorem 9], the following robust inverse inequality in $\widetilde{\mathcal{S}}_{p,\ell}(\Omega)$, analogous to the one-dimensional result (16), follows:

$$\|u_\ell\|_{\mathcal{A}} \leq 2\sqrt{6} h_\ell^{-1} \|u_\ell\|_{L^2(\Omega)} \qquad \forall u_\ell \in \widetilde{\mathcal{S}}_{p,\ell}(\Omega).$$

As in Section 4 for the one-dimensional case, we could set up a smoother based on the $\mathcal{A}$-orthogonal projection to an interior space $S^I_{p,\ell}(0,1) \otimes S^I_{p,\ell}(0,1)$ and prove, by completely analogous arguments, that the resulting multigrid method is robust. However, the realization of this smoother requires the discrete harmonic extension from a boundary layer of width p to the interior, which is too computationally expensive.

Instead we propose a slightly modified smoother which better exploits the tensor product structure of the underlying spaces. Using $A_\ell = K_\ell + M_\ell$, we obtain

$$\mathcal{A}_\ell \leq A_\ell \otimes M_\ell + M_\ell \otimes A_\ell.$$

As we have seen that $A_\ell \le cL_\ell$ for the choice of L_ℓ introduced in Section 4, the inverse inequality would be satisfied for the choice

$$L_\ell \otimes M_\ell + M_\ell \otimes L_\ell,$$

which can be expressed using $L_\ell = h_\ell^{-2} M_\ell + C_\ell$ from (20) as follows:

$$2h_\ell^{-2} M_\ell \otimes M_\ell + C_\ell \otimes M_\ell + M_\ell \otimes C_\ell$$

Instead we choose $\mathcal{L}_\ell$ to be a slight perturbation of that matrix, namely,

$$\mathcal{L}_\ell := h_\ell^{-2} M_\ell \otimes M_\ell + C_\ell \otimes M_\ell + M_\ell \otimes C_\ell = h_\ell^2 (L_\ell \otimes L_\ell - C_\ell \otimes C_\ell),$$

Thus, $\mathcal{L}_\ell$ is the sum of a tensor product matrix $h_\ell^2 L_\ell \otimes L_\ell$ and a low-rank correction $h_\ell^2 C_\ell \otimes C_\ell$. The following two lemmas show the assumptions for multigrid convergence when using the smoother based on this $\mathcal{L}_\ell$.

Lemma 11. *There is a constant c, independent of ℓ and p, such that $\mathcal{A}_\ell \le c\,\mathcal{L}_\ell$.*

Proof. The one-dimensional inverse inequality (18) in matrix formulation reads $A_\ell \le cL_\ell$. Using $L_\ell = h_\ell^{-2} M_\ell + C_\ell$, we obtain

$$\begin{aligned}\mathcal{A}_\ell &\le A_\ell \otimes M_\ell + M_\ell \otimes A_\ell \le c(L_\ell \otimes M_\ell + M_\ell \otimes L_\ell)\\ &= c(2h_\ell^{-2} M_\ell \otimes M_\ell + C_\ell \otimes M_\ell + M_\ell \otimes C_\ell)\\ &\le 2c(h_\ell^{-2} M_\ell \otimes M_\ell + C_\ell \otimes M_\ell + M_\ell \otimes C_\ell) = 2c\mathcal{L}_\ell.\end{aligned}$$

□

Lemma 12. *The approximation property*

$$\|(I - \mathcal{T}_{\ell-1})u_\ell\|_{\mathcal{L}_\ell} \le c\|u_\ell\|_{\mathcal{A}} \qquad \forall u_\ell \in \mathcal{S}_{p,\ell}(\Omega)$$

holds with a constant c which is independent of ℓ and p.

Proof. By an energy minimization argument, we know that $C_\ell \le A_\ell$ and therefore

$$\mathcal{L}_\ell \le h_\ell^{-2} \mathcal{M}_\ell + M_\ell \otimes A_\ell + A_\ell \otimes M_\ell.$$

Furthermore, since $M_\ell \otimes A_\ell + A_\ell \otimes M_\ell = \mathcal{A}_\ell + \mathcal{M}_\ell$ and $h_\ell \le 1$, we obtain

$$\mathcal{L}_\ell \le 2h_\ell^{-2} \mathcal{M}_\ell + \mathcal{A}_\ell.$$

Thus, using (22) and the stability of the $\mathcal{A}$-orthogonal projector $I - \mathcal{T}_{\ell-1}$, we have

$$\|(I - \mathcal{T}_{\ell-1})u_\ell\|_{\mathcal{L}_\ell}^2 \le 2\|(I - \mathcal{T}_{\ell-1})u_\ell\|_{h_\ell^{-2}\mathcal{M}_\ell}^2 + \|(I - \mathcal{T}_{\ell-1})u_\ell\|_{\mathcal{A}}^2 \le (2c+1)\|u_\ell\|_{\mathcal{A}}^2$$

for all $u_\ell \in \widetilde{S}_{p,\ell}(\Omega)$. □

We now immediately obtain the following convergence result.

Theorem 13. *Consider the smoother* (2) *with*

$$\mathcal{L}_\ell = h_\ell^2 (L_\ell \otimes L_\ell - C_\ell \otimes C_\ell),$$

where $L_\ell = h_\ell^{-2} M_\ell + C_\ell$ *and* $C_\ell = (I - T_\ell^I)^T A (I - T_\ell^I)$ *as in Section 4, and* $T_\ell^I : S_{p,\ell}(0,1) \to S_{p,\ell}^I(0,1)$ *is the A-orthogonal projector. Then, there is some damping parameter* $\tau > 0$ *and some choice* $\nu_0 > 0$*, both independent of* ℓ *and* p*, such that the two-grid method with* $\nu > \nu_0$ *smoothing steps converges with rate* $q = \nu_0/\nu$.

With the same choice of τ *and with* $\nu > 4\nu_0$*, also the W-cycle multigrid method with* $\nu/2$ *pre- and* $\nu/2$ *post-smoothing steps converges in the energy norm with convergence rate* $q = 2\nu_0/\nu$.

Proof. Lemma 11 and Lemma 12 show the assumptions (10) and (9) of Theorem 3, respectively. The statements then follow from Theorems 3 and 4. □

To obtain a quasi-optimal method, we need a way of inverting the matrix $\mathcal{L}_\ell$ efficiently. Grouping the degrees of freedom as in (19), we obtain

$$\mathcal{L}_\ell = h_\ell^2 L_\ell \otimes L_\ell - h_\ell^2 [P_\ell^T \otimes P_\ell^T][Q_\ell \otimes Q_\ell][P_\ell \otimes P_\ell] \tag{23}$$

with

$$P_\ell := \begin{pmatrix} I \\ 0 \end{pmatrix} \qquad \text{and} \qquad Q_\ell := A_{\Gamma\Gamma,\ell} - A_{I\Gamma,\ell}^T A_{II,\ell}^{-1} A_{I\Gamma,\ell}. \tag{24}$$

Exploiting the Kronecker product structure, $L_\ell \otimes L_\ell$ can be efficiently inverted using the ideas from [5]. The second term, $h_\ell^2 [P_\ell^T \otimes P_\ell^T][Q_\ell \otimes Q_\ell][P_\ell \otimes P_\ell]$, is a low-rank correction that lives only near the four corners of the domain. We can thus invert the smoother using the Sherman-Morrison-Woodbury formula and obtain

$$\mathcal{L}_\ell^{-1} = h_\ell^{-2} (I + ([L_\ell^{-1} P_\ell] \otimes [L_\ell^{-1} P_\ell]) \mathcal{R}_\ell^{-1} (P_\ell^T \otimes P_\ell^T))(L_\ell^{-1} \otimes L_\ell^{-1}),$$

where P_ℓ and Q_ℓ are defined as in (24),

$$\mathcal{R}_\ell := Q_\ell^{-1} \otimes Q_\ell^{-1} - W_\ell^{-1} \otimes W_\ell^{-1}, \tag{25}$$

$$W_\ell := Q_\ell + h_\ell^{-2} (M_{\Gamma\Gamma,\ell} - M_{I\Gamma,\ell}^T M_{II,\ell}^{-1} M_{I\Gamma,\ell}), \tag{26}$$

and $M_{II,\ell}$, $M_{I\Gamma,\ell}$ and $M_{\Gamma\Gamma,\ell}$ are the blocks of the mass matrix M_ℓ if the degrees of freedom are grouped as in (19).

6. Numerical realization and results

6.1. Numerical realization

As the proposed smoothing procedure appears rather complex, we give in Listing 6.1 pseudo code describing how the two-dimensional smoother based on

function smoother()

- **given matrices:** one-dimensional mass and stiffness matrices $M_\ell, A_\ell \in \mathbb{R}^{m_\ell \times m_\ell}$
- **input:** function value $\underline{u}_\ell^{(0,n)} \in \mathbb{R}^{m_\ell^2}$, corresponding residual $\underline{r}_\ell^{(0,n)} \in \mathbb{R}^{m_\ell^2}$
- Preparatory steps, only performed once:
  1. Compute the Cholesky factorization of $A_{II,\ell}$.
  2. Determine $Q_\ell \in \mathbb{R}^{2p \times 2p}$ as defined in (24).
  3. Compute the Cholesky factorization of $M_{II,\ell}$.
  4. Determine $W_\ell \in \mathbb{R}^{2p \times 2p}$ as defined in (26).
  5. Determine $\mathcal{R}_\ell \in \mathbb{R}^{4p^2 \times 4p^2}$ as defined in (25).
  6. Compute the Cholesky factorization of $\mathcal{R}_\ell$.
  7. Determine the sparse matrix $L_\ell \in \mathbb{R}^{m_\ell \times m_\ell}$ as defined in (20).
  8. Compute the Cholesky factorization of L_ℓ.
- For each smoothing step do:
  1. Determine $\underline{q}_\ell^{(0)} := h_\ell^{-2}(L_\ell^{-1} \otimes L_\ell^{-1})\underline{r}_\ell$.
  2. Determine $\underline{q}_\ell^{(1)} := (P_\ell^T \otimes P_\ell^T)\underline{q}_\ell^{(0)}$.
  3. Determine $\underline{q}_\ell^{(2)} := \mathcal{R}_\ell^{-1}\underline{q}_\ell^{(1)}$ using the Cholesky factorization of $\mathcal{R}_\ell$.
  4. Determine $\underline{q}_\ell := ([L_\ell^{-1}P_\ell] \otimes [L_\ell^{-1}P_\ell])\underline{q}_\ell^{(2)}$ using the Cholesky factorization of L_ℓ.
  5. Set $\underline{p}_\ell = \underline{q}_\ell^{(0)} + \underline{q}_\ell$.
  6. Update the function $\underline{u}_\ell^{(0,n+1)} := \underline{u}_\ell^{(0,n)} + \tau \underline{p}_\ell$.
  7. Update the residual $\underline{r}_\ell^{(0,n+1)} := \underline{r}_\ell^{(0,n)} - \tau \mathcal{A}_\ell \underline{p}_\ell$.
- **output:** function value $\underline{u}_\ell^{(0,n+1)}$, corresponding residual $\underline{r}_\ell^{(0,n+1)}$

Listing 1: Pseudo-code implementation for the smoother in two dimensions the matrix $\mathcal{L}_\ell$ as described in (23) can be implemented efficiently such that the overall multigrid method achieves optimal complexity.

The overall costs of the preparatory steps are $\mathcal{O}(m_\ell p^2 + p^6)$ floating point operations, where $m_\ell > p$ is the number of degrees of freedom in one dimension and m_ℓ^2 is the overall number of degrees of freedom.

Here, for the preparatory steps 1 and 3, $\mathcal{O}(m_\ell p^2)$ operations are required, as the dimension of the matrices $A_{II,\ell}$ and $M_{II,\ell}$ is $m_\ell - 2p = \mathcal{O}(m_\ell)$ and the bandwidth is $\mathcal{O}(p)$. For the preparatory steps 2 and 4, it is required to solve $\mathcal{O}(p)$ linear systems involving this factorization, which requires again $\mathcal{O}(m_\ell p^2)$ operations. The preparatory steps 5 and 6 just live on the 4 vertices of the square and require $\mathcal{O}(p^6)$ operations. The preparatory step 7 costs $\mathcal{O}(m_\ell p)$ operations just for adding A_ℓ and L_ℓ. The Cholesky factorization to be performed in the preparatory step 8 has – as in the steps 1 and 3 – a computational complexity of $\mathcal{O}(m_\ell p^2)$ operations.

The steps 1 and 4 of the smoother itself require $\mathcal{O}(m_\ell^2 p)$ operations each, if the tensor product structure $L_\ell^{-1} \otimes L_\ell^{-1} = (I \otimes L_\ell^{-1})(L_\ell^{-1} \otimes I)$ is used (see [5] for the algorithmic idea). Step 2 of the smoother requires $\mathcal{O}(m_\ell p^2)$ operations. In step 3, only $\mathcal{O}(p^4)$ operations are required as $\mathcal{R}_\ell$ is a dense $4p^2 \times 4p^2$-matrix. The steps 5 and 6, which are only adding up, can be completed with $\mathcal{O}(m_\ell^2)$ operations. Step 7 can be computed using the decomposition

$$\mathcal{A}_\ell = (I \otimes M_\ell)(K_\ell \otimes I) + (M_\ell \otimes I)(I \otimes K_\ell) + (M_\ell \otimes I)(I \otimes M_\ell) \tag{27}$$

with $\mathcal{O}(m_\ell^2 p)$ operations.

In summary, the preparatory steps require $\mathcal{O}(m_\ell p^2 + p^6)$ operations and the smoother itself requires $\mathcal{O}(m_\ell^2 p)$ operations. Furthermore, also the coarse-grid correction can be performed with $\mathcal{O}(m_\ell^2 p)$ operations, if the tensor product structure of the intergrid transfer matrices $\mathcal{I}_\ell^{\ell-1}$ and $\mathcal{I}_{\ell-1}^{\ell}$ is used. The overall costs are therefore $\mathcal{O}(m_\ell^2 p + p^6)$ or, assuming $p^5 \leq m_\ell^2$, $\mathcal{O}(m_\ell^2 p)$ operations. The method can therefore be called asymptotically optimal, since the multigrid solver requires $\mathcal{O}(m_\ell^2 p)$ operations in total, which is the same effort as for multiplication with $\mathcal{A}_\ell$.

In the case of variable coefficients or geometry transformations, $\mathcal{A}_\ell$ is no longer a sum of tensor-product matrices. In this case, the computation of the residual of the original problem requires $\mathcal{O}(m_\ell^2 p^2)$ operations, while the application of the multigrid method as a preconditioner can still be performed with $\mathcal{O}(m_\ell^2 p)$ operations.

6.2. Experimental results

As a numerical example, we solve the model problem from Section 2.2 with the right-hand side

$$f(\mathbf{x}) = d\pi^2 \prod_{j=1}^{d} \sin(\pi(x_j + \tfrac{1}{2}))$$

and homogeneous Neumann boundary conditions on the domain $\Omega = (0,1)^d$, $d = 1, 2$. We perform a (tensor product) B-spline discretization using uniformly sized knot spans and maximum-continuity splines for varying spline degrees p. We refer to the coarse discretization with only a single interval as level $\ell = 0$ and perform uniform, dyadic refinement to obtain the finer discretization levels ℓ with $2^{\ell d}$ elements (intervals or quadrilaterals).

We set up a V-cycle multigrid method according to the framework established in Section 3 and using the proposed smoother (20) for the one-dimensional domain and (23) for the two dimensional domain. We always use one pre- and one post-smoothing step. The damping parameter τ was chosen as $\tau = 0.14$ for $d = 1$ and $\tau = 0.08$ for $d = 2$. In the one-dimensional case, we apply the damping factor only to the mass component of the smoother. This slightly improves the iteration numbers but is non-essential.

Due to the requirement that the number of intervals exceed p so that the space $S^I_{p,\ell}(0,1)$ is nonempty and the construction of the smoother is valid, we cannot always use $\ell = 0$ as the coarse grid for the multigrid method.

We perform two-grid iteration until the Euclidean norm of the initial residual is reduced by a factor of 10^{-8}. The iteration numbers using varying spline degrees p as well as varying fine grid levels ℓ for the one-dimensional domain are given in Table 1. Here we always used $\ell = 5$ as the coarse grid since this is an admissible choice for all tested p. We observe that the iteration numbers are robust with respect to the grid size, the number of levels, and the spline degree p.

In the two-dimensional experiments, for every degree p, we determine the coarsest level on which $S^I_{p,\ell}(0,1)$ is nonempty and use the next coarser level as the coarsest grid for the multigrid method. The resulting iteration numbers are displayed in Table 2. Here, the bottom-most iteration number in each column corresponds to a two-grid method, the next higher one to a three-grid method, and so on. Again, the iteration numbers remain uniformly bounded with respect to all discretization parameters, although they are much higher than in the one-dimensional case. To mitigate this, we can solve the problem with Conjugate Gradient (CG) iteration preconditioned with one multigrid V-cycle. As the resulting iteration numbers in Table 3, now only for $\ell = 7$, show, this leads to a significant speedup while maintaining robustness.

		p														
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	12	23	20	20	20	20	20	20	20	20	19	19	19	19	18	18
ℓ	11	23	20	20	20	20	20	20	20	19	19	19	19	18	19	18
	10	23	20	20	20	20	20	20	19	19	19	19	18	17	17	17

Table 1: Multigrid iteration numbers in 1D for varying fine-grid levels and spline degrees.

7. Conclusions and outlook

We have analyzed in detail the convergence properties of a geometric multigrid method for an isogeometric model problem using B-splines. In a first step,

								p							
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	7	86	88	99	102	99	100	99	98	97	96	94	95	93	92	92
	6	84	89	101	104	100	101	100	97	97	96	94	94	94	93	92
	5	83	92	103	103	100	100	101	97	97	96	94	93	94	91	91
ℓ	4	66	95	104	105	102	100	99	96	96	95	94	92	92	91	91
	3	45	97	105	107	103	101	101	-	-	-	-	-	-	-	-
	2	32	97	114	-	-	-	-	-	-	-	-	-	-	-	-
	1	32	-	-	-	-	-	-	-	-	-	-	-	-	-	-

Table 2: Multigrid iteration numbers in 2D for varying fine-grid levels and spline degrees.

p	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\ell = 7$	21	21	23	23	23	22	23	22	22	22	21	21	21	21	21

Table 3: Iteration numbers in 2D for CG preconditioned with V-cycle, varying p.

we discussed one-dimensional domains. Based on the obtained insights, in particular on the importance of boundary effects on the convergence rate, we then proposed a boundary-corrected mass-Richardson smoother. We have proved that this new smoother yields convergence rates which are robust with respect to the spline degree, a result which is not attainable with classical [14] or purely mass-based smoothers [12].

Exploiting the tensor product structure of spline spaces commonly used in IgA, we extended the construction of the smoother to the two-dimensional setting and again proved robust convergence for this case. We have shown how the proposed smoother can be efficiently realized such that the overall multigrid method has quasi-optimal complexity. Although we have restricted ourselves to simple tensor product domains, our technique is easily extended to problems with non-singular geometry mappings as discussed in Remark 1.

The extension of this approach to three or more dimensions remains open. In particular, the representation of the smoother as the Kronecker product of one-dimensional smoothers plus some low-rank correction as in (23) encounters some difficulties in this case. Therefore, the construction of robust and efficient smoothers for the three- and higher-dimensional cases is left as future work.

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Appendix

Our aim in this section is to prove Theorem 6. First, however, we show a refined version of the main approximation result from [16].

Theorem 14. *Let $\ell \in \mathbb{N}_0$ and $p \in \mathbb{N}$. Then*

$$\|(I - \widetilde{\Pi}_\ell)u\|_{L^2(\Omega)} \leq \sqrt{2} h_\ell |u|_{H^1(\Omega)} \qquad \forall u \in H^1(\Omega),$$

where $\widetilde{\Pi}_\ell : H^1(\Omega) \to \widetilde{S}_{p,\ell}(\Omega)$ is the $H^1_\circ(\Omega)$-orthogonal projector and

$$(u, v)_{H^1_\circ(\Omega)} := (\nabla u, \nabla v)_{L^2(\Omega)} + \frac{1}{|\Omega|} \left(\int_\Omega u(x) \mathrm{d}x \right) \left(\int_\Omega v(x) \mathrm{d}x \right).$$

Proof. The proof is a slightly improved version of the proof of [16, Theorem 1]. There, for any fixed $u \in H^1(\Omega)$, a function $u_\ell \in \widetilde{S}_{p,\ell}(\Omega)$ was constructed which satisfies the approximation error estimate

$$\|u - u_\ell\|_{L^2(\Omega)} \leq \sqrt{2} h_\ell |u|_{H^1(\Omega)}.$$

(The space $\widetilde{S}_{p,\ell}(\Omega)$ coincides with the space $\widetilde{S}_{p,h_\ell}(\Omega)$ in [16].) Now, it remains to show that u_ℓ coincides with $\widetilde{\Pi}_\ell u$, i.e., that u_ℓ is the $H^1_\circ(\Omega)$-orthogonal projection of u to the space $\widetilde{S}_{p,\ell}(\Omega)$. By definition, this means that we have to show that

$$(u - u_\ell, \widetilde{u}_\ell)_{H^1_\circ(\Omega)} = 0 \qquad \forall \widetilde{u}_\ell \in \widetilde{S}_{p,\ell}(\Omega). \tag{28}$$

Now, let $\widehat{S}_{p,\ell}(-1,1)$ be the space of periodic splines on $(-1,1)$ with degree p and grid size h_ℓ (cf. [16, Definition 2]). Note that, by construction in [16, Theorem 1], u_ℓ is the restriction of w_ℓ to $(0,1)$, where w_ℓ is the $H^1_\circ$-orthogonal projection of $w(x) := u(|x|)$ to $\widehat{S}_{p,\ell}(-1,1)$. Define $\widetilde{w}_\ell \in \widehat{S}_{p,\ell}(-1,1)$ by $\widetilde{w}_\ell(x) := \widetilde{u}_\ell(|x|)$ and observe $(w - w_\ell, \widetilde{w}_\ell)_{H^1_\circ(-1,1)} = 2(u - u_\ell, \widetilde{u}_\ell)_{H^1_\circ(0,1)}$. As w_ℓ is by construction the $H^1_\circ$-orthogonal projection of w, we have $(w - w_\ell, \widetilde{w}_\ell)_{H^1_\circ(0,1)} = 0$. This shows (28) and therefore $u_\ell = \widetilde{\Pi}_\ell u$. □

The error estimate for the A-orthogonal, i.e., the $H^1(\Omega)$-orthogonal, projector $\widetilde{T}_\ell$ now follows immediately by a perturbation argument.

Proof (of Theorem 6). Define, for $u, v \in H^1(\Omega)$, the scalar product

$$(u, v)_B := (u, v)_A - (u, v)_{H^1_\circ(\Omega)} = (u, v)_{L^2(\Omega)} - \frac{1}{|\Omega|} \left(\int_\Omega u \mathrm{d}x \right) \left(\int_\Omega v \mathrm{d}x \right)$$

and observe that the Cauchy-Schwarz inequality implies $0 \leq (v, v)_B \leq (v, v)_{L^2(\Omega)}$ for all $v \in H^1(\Omega)$. Let $u \in H^1(\Omega)$ arbitrary. Orthogonality implies

$$\begin{aligned} ((I - \widetilde{\Pi}_\ell)u, w_\ell)_{H^1_\circ(\Omega)} &= 0 && \forall w_\ell \in \widetilde{S}_{p,\ell}(\Omega), \\ ((I - \widetilde{T}_\ell)u, w_\ell)_A &= 0 && \forall w_\ell \in \widetilde{S}_{p,\ell}(\Omega). \end{aligned} \tag{29}$$

The first of these two equations implies

$$((I - \widetilde{\Pi}_\ell)u, w_\ell)_A = ((I - \widetilde{\Pi}_\ell)u, w_\ell)_B \qquad \forall w_\ell \in \widetilde{S}_{p,\ell}(\Omega),$$

and subtracting (29) yields

$$((\widetilde{T}_\ell - \widetilde{\Pi}_\ell)u, w_\ell)_A = ((I - \widetilde{\Pi}_\ell)u, w_\ell)_B \qquad \forall w_\ell \in \widetilde{S}_{p,\ell}(\Omega).$$

With the choice $w_\ell := (\widetilde{T}_\ell - \widetilde{\Pi}_\ell)u$ as well as $\|\cdot\|_A = \|\cdot\|_{H^1(\Omega)}$ and the Cauchy-Schwarz inequality, we obtain

$$\|(\widetilde{T}_\ell - \widetilde{\Pi}_\ell)u\|^2_{H^1(\Omega)} \le \|(I - \widetilde{\Pi}_\ell)u\|_B \|(\widetilde{T}_\ell - \widetilde{\Pi}_\ell)u\|_B,$$

where $\|\cdot\|_B := (\cdot,\cdot)_B^{1/2}$. Using $\|\cdot\|_B \le \|\cdot\|_{L^2(\Omega)} \le \|\cdot\|_{H^1(\Omega)}$, it follows that $\|(\widetilde{T}_\ell - \widetilde{\Pi}_\ell)u\|_{L^2(\Omega)} \le \|(I - \widetilde{\Pi}_\ell)u\|_{L^2(\Omega)}$. Thus, using the triangle inequality and Theorem 14 we obtain

$$\begin{aligned}\|(I - \widetilde{T}_\ell)u\|_{L^2(\Omega)} &\le \|(I - \widetilde{\Pi}_\ell)u\|_{L^2(\Omega)} + \|(\widetilde{T}_\ell - \widetilde{\Pi}_\ell)u\|_{L^2(\Omega)} \\ &\le 2\|(I - \widetilde{\Pi}_\ell)u\|_{L^2(\Omega)} \le 2\sqrt{2}h_\ell\|u\|_{H^1(\Omega)}.\end{aligned}$$

□

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