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Strongly stable bases for adaptively refined multilevel spline spaces

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Abstract The problem of constructing a normalized hierarchical basis for adaptively refined spline spaces is addressed. Multilevel representations are defined in terms of a hierarchy of basis functions, reflecting different levels of refinement. When the hierarchical model is constructed by considering an underlying sequence of bases $\{\Gamma^\ell\}_{\ell=0,\ldots,N-1}$ with properties analogous to classical tensor–product B–splines, we can define a set of locally supported basis functions that form a partition of unity and possess the property of coefficient preservation, i.e., they preserve the coefficients of functions represented with respect to one of the bases Γ^ℓ. Our construction relies on a certain *truncation* procedure, which eliminates the contributions of functions from finer levels in the hierarchy to coarser level ones. Consequently, the support of the original basis functions defined on coarse grids is possibly reduced according to finer levels in the hierarchy. This truncation mechanism not only decreases the overlapping of basis supports, but it also guarantees *strong stability* of the construction. In addition to presenting the theory for the general framework, we apply it to hierarchically refined tensor–product spline spaces, under certain reasonable assumptions on the given knot configuration.

Keywords hierarchical splines · truncated hierarchical basis · partition of unity · local refinement · stability

1 Introduction

The choice of the basis used to define a representation model is of extreme importance because it necessarily influences both the geometrical and numerical characteristics of the resulting mathematical technology. Recent studies connected to isogeometric analysis [12] are devoted to the investigation of appropriate adaptive

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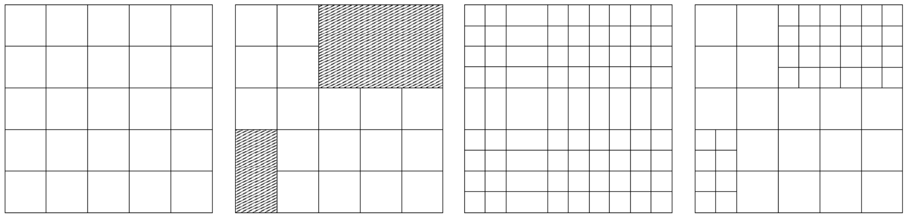


Fig. 1 Given an initial tensor–product representation (a), an error estimator indicates regions of the mesh which require further refinement (b). The tensor–product structure necessarily implies a propagation of the refinement (c). Adaptive splines, instead, should provide a proper local control of the refinement procedure (d).

spline bases which provide a well–localized mesh refinement, see, e.g., [1, 6, 7, 13, 18, 25, 26]. The tensor–product structure underlying the B–spline/NURBS model originally considered in [12] necessarily forces a global refinement of the geometry, as shown in Figure 1.

The hierarchical spline model provides a natural strategy to guarantee the locality of the refinement. Hierarchical representations are defined in terms of a hierarchy of locally refined meshes, reflecting different levels of refinement. This idea was first introduced in [9] by using a sequence of overlays to define hierarchical spline surfaces. The possibility of restricting the refinement to specific regions of the mesh allows an effective local surface editing. A hierarchical B–spline basis was then developed in [14].

The hierarchical B–spline basis of [14], however, suffers from three major problems.

- First, the basis does not form a partition of unity. It is possible to address this problem by appropriately rescaling the basis functions, but additional assumptions are then needed to guarantee non–negativity of the scaling factors.
- Second, the support of the basis functions is relatively large. Consequently, when applying the hierarchical construction without enforcing additional restrictions on the refinement strategies, the number of basis functions which act on the same region of the parametric domain may easily increase. In particular, this happens when several refinement steps are needed to describe detailed local features.
- Third, the hierarchical B–spline basis is only weakly stable[1] with respect to the supremum norm. The absence of a stronger form of stability is implied by the missing partition of unity and the relatively large support of the hierarchical basis.

Different kinds of hierarchical splines defined over triangulations have also been considered, see for example [23, 27], in order to investigate the flexibility of a hierarchical refinement procedure for more general mesh configurations. For hierarchical splines over Powell–Sabin refined triangulations, strong stability of a special basis was established in [23].

[1] Weak stability of a basis means that the associated stability constants have at most a polynomial growth in the number of hierarchical levels.

In our paper we propose a general framework for the construction of a new basis which overcomes the first and the third problem of the classical hierarchical basis and also shows an improved behaviour with respect to the second one. More precisely, we consider adaptively refined multilevel spline spaces which are constructed by considering an underlying basis with properties analogous to classical normalized B–splines, such as *local linear independence*, *compact support*, *non–negativity*, *partition of unity*. We propose a new basis possessing the following properties.

- We define a linearly independent set of locally supported basis functions for the multilevel spline space that forms a convex partition of unity. To this end, we propose a certain *truncation procedure*, which is applied to hierarchical basis functions to eliminate the contributions from higher levels in the hierarchy. The corresponding basis is denoted as *truncated hierarchical spline basis*. The bases constructed in [10] and [23] fit in this general framework. The presented truncation mechanism preserves the linear independence and non–negativity properties of standard, i.e., non–truncated, hierarchical bases.
- The truncation gives basis functions with the same or smaller supports. Consequently, the overlap between basis functions associated with different levels in the hierarchy is effectively reduced. These properties lead to sparser systems when this kind of basis is used in classical approximation problems, see [10].
- We prove that the truncation procedure preserves the corresponding coefficients of a function represented with respect to the underlying basis considered at any certain level. This property implies the partition of unity and the preservation of the original Greville points.
- When assuming an underlying locally stable basis, we show that this preservation of coefficients can be properly used to prove the *strong stability* of the construction with respect to the supremum norm. This means that the constants to be considered in the stability analysis of the basis do not depend on the number of hierarchical levels. We also apply this general theory to the context of truncated hierarchical B–splines (THB–splines), introduced in [10]. Assuming very mild conditions on the knot configurations, we show that the stability properties of the THB–spline basis are improved with respect to the standard — i.e., non–truncated and *weakly stable* — hierarchical B–spline case originally considered in [14, 15].

The remainder of this paper is organized as follows. Sections 2 and 3 describe a general framework of adaptively refined spline spaces and present a recursive definition for a hierarchical basis, assuming some specific properties for the choice of the underlying spaces to be considered in the multilevel construction. In Section 4 we introduce an alternative basis, based on the above mentioned truncation procedure, together with its main properties. The preservation of coefficients and its implications are subsequently discussed in Section 5. Section 6 introduces the stability analysis by considering the case of a single mesh element and some remarks regarding the tensor–product construction. Section 7 discusses the stability of truncated bases in the general setting previously introduced. To illustrate the truncation mechanism, multivariate truncated hierarchical B–splines and their stability analysis are detailed throughout the text as a case study. We conclude in Section 8 by summarizing the main results of this work. Finally, Appendix A is devoted to specific bounds for the number of truncated basis functions acting on any point in the domain, by considering some restrictions on the configuration of the hierarchical sequence of subdomains. The local stability of univariate B–splines and a counterexample for strong stability of the classical (i.e., non–truncated) hierarchical construction are deferred to Appendix B and C, respectively.

2 Preliminaries

Given a bounded open domain $\Omega \subseteq \mathbb{R}^n$, we consider an infinite sequence of nested linear spaces

$$V^0 \subset V^1 \subset V^2 \subset \ldots$$

Each space V^ℓ is assumed to be a finite–dimensional subspace of the space $\mathcal{C}(\Omega,\mathbb{R})$ of all continuous n–variate functions on Ω with values in $\mathbb{R}$. The index ℓ of a space V^ℓ will be called its *level*. We assume that each space V^ℓ is spanned by a finite basis Γ^ℓ with the following properties.

(A1) The functions in Γ^ℓ are *locally linearly independent:* for any open subdomain $B \subseteq \Omega$, those functions in $\{\gamma\big|_B \colon \gamma \in \Gamma^\ell\}$ that do not vanish identically on B are linearly independent, where $f\big|_B \in \mathcal{C}(B,\mathbb{R})$ denotes the restriction of $f \in \mathcal{C}(\Omega,\mathbb{R})$ to B.
(A2) The functions in Γ^ℓ have *local and compact support:* $\operatorname{supp}\gamma \subset \Omega$ for all $\gamma \in \Gamma^\ell$. Moreover, for each basis function $\gamma \in \Gamma^\ell$, the boundary of the support consists of a finite number of smooth arcs.
(A3) The functions in Γ^ℓ are *non–negative.*
(A4) The functions in Γ^ℓ form a *partition of unity.*
(A5) The *two–scale relations* between adjacent bases have *only non–negative coefficients.* More precisely, for any level ℓ, each basis function in Γ^ℓ can be expressed as a linear combination of basis functions in $\Gamma^{\ell+1}$ with non–negative coefficients.

Together, requirements (A3) and (A4) imply the *convex hull* property. The local linear independence (A1) implies that the support of any function $\gamma^{\ell+1}$ in $\Gamma^{\ell+1}$ which contributes to the representation of a function γ^ℓ in Γ^ℓ is contained in the support of γ^ℓ.

Several families of basis functions and their corresponding function spaces satisfy assumptions (A1–A5).

- From the classical univariate spline theory, it is known that B–splines [2, 22], NURBS [19] and many generalized B–splines [4, 22] satisfy them. Moreover, they also hold for tensor–products of these univariate splines which are defined on Cartesian grids.
- The above assumptions are satisfied for type–I box splines [16] which are bivariate functions defined on a three–directional grid. Note that linear independence and local linear independence are equivalent for box splines [3]. Hence, (A1) is satisfied.
- Finally, these assumptions are also satisfied for the bivariate Powell–Sabin B–splines with triadic refinement studied in [23, 24].

Motivated by these similarities between various spline spaces, we shall say that Γ^ℓ are *spline bases* and V^ℓ are *spline spaces*, provided that assumptions (A1–A5) are valid.

We denote with $\overline{B}$ the closure of an open set $B \subset \mathbb{R}^n$ and with $\partial\overline{B}$ its boundary. The boundaries $\partial\,\overline{\operatorname{supp}\gamma}$ of the supports of all basis functions $\gamma \in \Gamma^\ell$ subdivide the domain Ω into a number of connected so–called *cells* (or *patches*). More precisely, we denote with Π^ℓ the set of connected components of

$$\Omega \setminus \bigcup_{\gamma\in\Gamma^\ell} \partial\,\overline{\operatorname{supp}\gamma}. \tag{1}$$

It satisfies

$$\overline{\Omega} = \overline{\bigcup_{\pi\in\Pi^\ell} \pi} \quad \text{and} \quad \pi \cap \pi' = \emptyset \quad \text{for all} \quad \pi, \pi' \in \Pi^\ell.$$

Note also that the cells are either contained or not contained in the supports of basis functions,

$$\forall\gamma \in \Gamma^\ell, \forall\pi \in \Pi^\ell : \; \pi \cap \operatorname{supp}\gamma \neq \emptyset \;\Rightarrow\; \pi \subseteq \operatorname{supp}\gamma.$$

We further know that for each cell $\pi \in \Pi^{\ell+1}$ of level $\ell+1$ there exists a cell $\pi' \in \Pi^\ell$ of level ℓ such that $\pi \subseteq \pi'$.[2]

Throughout the text, we shall use the standard univariate and multivariate tensor–product normalized B–spline basis as case study to illustrate different constructions and properties. This will be done in the frame of Examples 1 and 2 consisting of several parts each. For a third example, we refer to [23] where the hierarchical setting, the truncation procedure and the stability analysis have been elaborated for Powell–Sabin B–splines with triadic refinement.[3]

Example 1: univariate B–splines (part 1) Univariate B–splines may be used as background basis for the hierarchical construction. We consider spline functions of a certain degree d which are defined by knot sequences $T^\ell = (t_i^\ell)_{i=0,\ldots,m^\ell}$. These knot sequences contain non–decreasing real values so that the multiplicities μ of the inner knot values satisfy

$$0 \le \mu(T^\ell, t) \le d + 1.$$

Here, $\mu(T^\ell, t)$ denotes the multiplicity of t in T^ℓ. Observe that $\mu(T^\ell, t) = 0$ if t is not a knot in T^ℓ. To guarantee the nested nature of the spline spaces, these knot sequences are also assumed to be nested,

$$\mu(T^{\ell+1}, t) \ge \mu(T^\ell, t).$$

The domain Ω may be chosen as any open real interval. For each level ℓ, let Γ^ℓ be the restriction of the B–spline basis of degree d to Ω, and we denote with V^ℓ the spline space spanned by Γ^ℓ. For each level ℓ, the set of cells Π^ℓ consists of all non–empty knot spans which are contained in Ω,

$$\Pi^\ell = \{\,]t_r^\ell, t_{r+1}^\ell[\,:\; t_d^0 \le t_r^\ell \neq t_{r+1}^\ell \le t_{m^0-d}^0,\; r = 0, \ldots, m^\ell - 1\}. \tag{2}$$

We may observe that properties (A1–A5) are satisfied according to the classical spline theory [2, 22].

[2] This can be shown as follows. Suppose there does not exist a cell π' of level ℓ such that $\pi \subseteq \pi'$, then there must be at least one coarse basis function of level ℓ whose support has a boundary crossing through π. This means that the coarse basis function cannot be represented in terms of finer basis functions of level $\ell + 1$ because all finer basis functions have at least support over the entire cell π or no support at all over π. This is in contradiction with the assumptions (A1–A5).

[3] Note that the truncated hierarchical basis is called quasi–hierarchical basis in [23, 24].

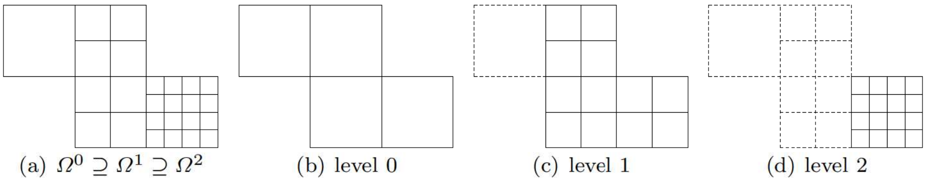


Fig. 2 An example of a 3–level subdomain hierarchy (a) for tensor–product bivariate splines. The subdomains Ω^0 (b), Ω^1 (c), and Ω^2 (d) are also shown (solid lines).

Example 2: tensor–product B–splines (part 1) In order to define multivariate tensor–product spaces, we consider n knot sequences $T_1^\ell, \ldots, T_n^\ell$ with the same properties as T^ℓ in the above example. Each knot vector T_i^ℓ defines a B–spline basis Γ_i^ℓ which spans the spline space V_i^ℓ. The basis Γ^ℓ defined as the tensor–product of $\Gamma_1^\ell, \Gamma_2^\ell, \ldots \Gamma_n^\ell$ spans the tensor–product space $V^\ell = V_1^\ell \otimes V_2^\ell \otimes \ldots \otimes V_n^\ell$. The domain Ω is the Cartesian product of the domains of the univariate spline spaces. For each level ℓ, the set of cells Π^ℓ consists of Cartesian products of intervals with respect to the n dimensions. It is well–known that properties (A1–A5) which characterize the univariate case are preserved by the tensor–product construction [22].

3 Multilevel spline spaces

In addition to the spaces V^ℓ and bases Γ^ℓ we will consider a sequence of nested open subdomains

$$\Omega^0 \supseteq \Omega^1 \supseteq \Omega^2 \supseteq \cdots$$

with $\Omega^0 \subseteq \Omega$, which we will call a *subdomain hierarchy*. Each Ω^ℓ represents the region selected to be refined at level ℓ. We assume that the closure of each subdomain Ω^ℓ coincides with the closure of a collection of cells $\pi \in \Pi^\ell$ of level ℓ. We also assume that there exists an integer N such that

$$\Omega^N = \emptyset.$$

The value of N represents the *depth* of the subdomain hierarchy. Thus, for a given subdomain hierarchy with depth N, only the first N spline spaces V^ℓ will be used for constructing the multilevel spline space.

Remark 1 The initial domain Ω has been introduced in order to allow for subdomains Ω^0 that are not just Cartesian products of intervals in the tensor–product spline case, see, e.g. the domain in Figure 2.

Now, given nested spline spaces V^ℓ with spline bases Γ^ℓ and nested subdomains Ω^ℓ, the construction of a hierarchical basis can be described as follows. We start by selecting all the basis functions in Γ^0 whose support overlaps Ω^0 and is not entirely contained in Ω^1. The spline hierarchy is then completed by adding from each basis Γ^ℓ, for $\ell = 1, \ldots, N-1$, the subset of basis functions whose support is completely contained in Ω^ℓ and overlaps with $D^\ell = \Omega^\ell \setminus \Omega^{\ell+1}$. This selection procedure is summarized in Definition 2, where we modify the usual support definition to

$$\operatorname{supp}^0 f = \operatorname{supp} f \cap \Omega^0.$$

Definition 2 *Given a subdomain hierarchy* $(\Omega^\ell)_{\ell=0,\dots,N-1}$, *the* hierarchical spline basis $\mathcal{H}$ *is recursively generated by the following algorithm.*

(I) Initialization: $\mathcal{H}^0 = \{\gamma \in \Gamma^0 : \operatorname{supp}^0 \gamma \neq \emptyset\}$.
(II) Recurrence: $\mathcal{H}^{\ell+1} = \mathcal{H}_A^{\ell+1} \cup \mathcal{H}_B^{\ell+1}$, *for* $\ell = 0,\dots,N-2$, *where*

$$\mathcal{H}_A^{\ell+1} = \left\{\gamma \in \mathcal{H}^\ell : \operatorname{supp}^0 \gamma \not\subseteq \Omega^{\ell+1}\right\},$$
$$\mathcal{H}_B^{\ell+1} = \left\{\gamma \in \Gamma^{\ell+1} : \operatorname{supp}^0 \gamma \subseteq \Omega^{\ell+1}\right\}.$$

(III) $\mathcal{H} = \mathcal{H}^{N-1}$.

We say that $S = \operatorname{span} \mathcal{H}$ *is the* multilevel spline space *which is determined by the subdomain hierarchy* $(\Omega^\ell)_{\ell=0,\dots,N-1}$.

Remark 3 In the above definition we construct the hierarchical basis by means of Γ^ℓ, $\ell = 0,\dots,N-1$, which are full spline bases defined over the entire domain Ω. Of course, in a practical setting, it is sufficient to take into account only a selection of basis functions in Γ^ℓ that possibly overlap with the subdomain Ω^ℓ.

Remark 4 The assumptions (A1–A5) on the bases Γ^ℓ allow us to choose a sequence of spaces V^ℓ of *different degrees*, as long as the considered spaces form a nested sequence. Hence, the hierarchical framework is not only confined to do adaptive h-refinement, but also p-refinement and k-refinement are possible.

The key properties of the hierarchical basis constructed according to Definition 2 are summarized by the following proposition.

Proposition 5 *By assuming that properties (A1–A5) hold for the bases* Γ^ℓ, *the hierarchical basis* $\mathcal{H}$ *possesses the following properties:*

(H1) linear independence: $\sum_{\gamma\in\mathcal{H}} c_\gamma \gamma = 0 \Leftrightarrow c_\gamma = 0,\ \forall \gamma \in \mathcal{H}$;
(H2) non–negativity: $\forall \gamma \in \mathcal{H},\ \gamma \geq 0$;
(H3) nested nature of the spline spaces: $\operatorname{span} \mathcal{H}^\ell \subseteq \operatorname{span} \mathcal{H}^{\ell+1}$ *for* $\ell = 0,\dots,N-2$.

Since we started by considering non–negative bases Γ^ℓ, as described in (A3), the non–negativity (H2) of the hierarchical basis $\mathcal{H}$ follows directly by definition. Properties (H1) and (H3) can be proved by generalizing the corresponding proofs for bivariate hierarchical B–splines, see Lemmas 2 and 3 in [25].

Next we consider two multilevel spline spaces $S = \operatorname{span} \mathcal{H}$ and $\hat{S} = \operatorname{span} \hat{\mathcal{H}}$ constructed by using the same underlying bases Γ^ℓ over two different subdomain hierarchies $(\Omega^\ell)_{\ell=0,\dots,N-1}$ and $(\hat{\Omega}^\ell)_{\ell=0,\dots,\hat{N}-1}$, with $N \leq \hat{N}$. If the second domain sequence enlarges the first one, then this property is inherited by the corresponding multilevel spline spaces.

Proposition 6 *If one of two sequences of nested subdomains is obtained by enlarging the other one while keeping the biggest subdomain, i.e.,* $\Omega^0 = \hat{\Omega}^0$ *and* $\Omega^\ell \subseteq \hat{\Omega}^\ell$ *for* $\ell = 1,\dots,N-1$, *then* $S \subseteq \hat{S}$.

Proof We denote with $S^\ell = \operatorname{span} \mathcal{H}^\ell$ and $\hat{S}^\ell = \operatorname{span} \hat{\mathcal{H}}^\ell$ the multilevel spaces spanned by the intermediate hierarchical bases in Definition 2. Thus, $S = S^{N-1}$ and $\hat{S} = \hat{S}^{\hat{N}-1}$.

Clearly, the recurrence (II) in Definition 2 gives $\mathcal{H}_B^{\ell+1} \subseteq \mathcal{H}^{\ell+1}$. By combining this with property (H3) of Proposition 5 we conclude that

$$\operatorname{span}(S^\ell \cup \mathcal{H}_B^{\ell+1}) \subseteq S^{\ell+1}. \tag{3}$$

On the other hand, using again the definition of the $\mathcal{H}^{\ell+1}$ we obtain

$$S^{\ell+1} = \operatorname{span} \mathcal{H}^{\ell+1} = \operatorname{span}(\mathcal{H}_A^{\ell+1} \cup \mathcal{H}_B^{\ell+1}) \subseteq \operatorname{span}(\mathcal{H}^\ell \cup \mathcal{H}_B^{\ell+1}) = \operatorname{span}(S^\ell \cup \mathcal{H}_B^{\ell+1}). \tag{4}$$

By combining these two observations we conclude that, for $\ell = 0, \ldots, N-2$,

$$S^{\ell+1} = \operatorname{span}(S^\ell \cup \mathcal{H}_B^{\ell+1}), \quad \text{and similarly} \quad \hat{S}^{\ell+1} = \operatorname{span}(\hat{S}^\ell \cup \hat{\mathcal{H}}_B^{\ell+1}). \tag{5}$$

When considering two sequences of nested subdomains, the relation $\Omega^{\ell+1} \subseteq \hat{\Omega}^{\ell+1}$ implies that more finer basis functions from $\Gamma^{\ell+1}$ are included in $\hat{\mathcal{H}}_B^{\ell+1}$ than in $\mathcal{H}_B^{\ell+1}$, i.e.

$$\mathcal{H}_B^{\ell+1} \subseteq \hat{\mathcal{H}}_B^{\ell+1}. \tag{6}$$

After observing that $\mathcal{H}^0 = \hat{\mathcal{H}}^0$ and $S^0 = \hat{S}^0$ due to $\Omega^0 = \hat{\Omega}^0$, we can now prove

$$S^{\ell+1} = \operatorname{span}(S^\ell \cup \mathcal{H}_B^{\ell+1}) \quad \subseteq \quad \hat{S}^{\ell+1} = \operatorname{span}(\hat{S}^\ell \cup \hat{\mathcal{H}}_B^{\ell+1}), \quad \ell = 0, \ldots, N-2,$$

by induction over ℓ. In particular, this means that $S = S^{N-1} \subseteq \hat{S}^{N-1}$. When $N = \hat{N}$, $\hat{S}^{N-1} = \hat{S}^{\hat{N}-1}$, and then the proof is complete. If $N < \hat{N}$ instead, in view of property (H3), we also have

$$\hat{S}^\ell \subseteq \hat{S}^{\ell+1}, \quad \ell = N-1, \ldots, \hat{N}-2.$$

Finally, we obtain

$$S = S^{N-1} \subseteq \hat{S}^{N-1} \subseteq \ldots \subseteq \hat{S}^{\hat{N}-1} = \hat{S}.$$

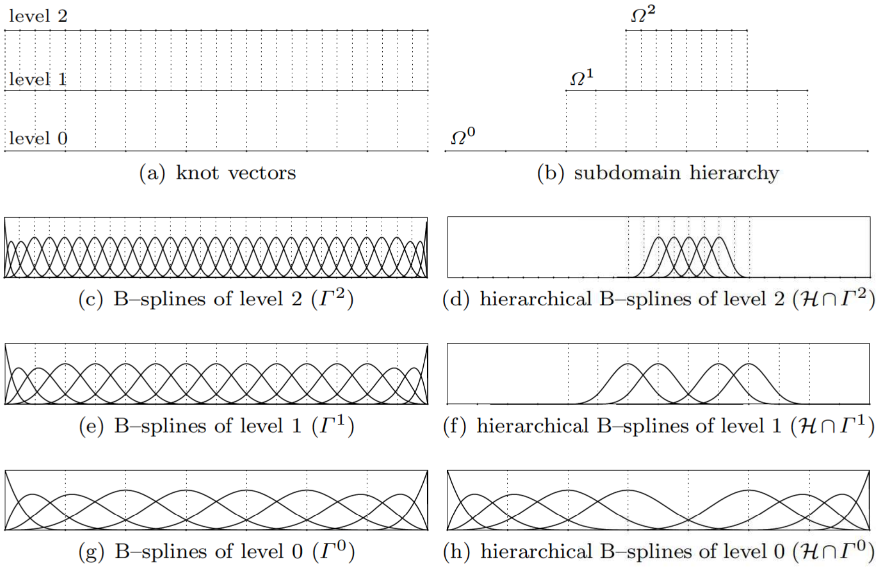
□

The partition of unity property (A4) is not inherited by the hierarchical basis. A simple way to recover the partition of unity property is by defining a *weighted basis* $\mathcal{W}$, by simply applying a suitable scaling to the hierarchical basis functions, namely

$$\mathcal{W} = \left\{ \omega = w_\gamma \gamma : \gamma \in \mathcal{H} \wedge 1 = \textstyle\sum_{\gamma \in \mathcal{H}} w_\gamma \gamma \right\}.$$

In view of (A4), we have that $1 \in V^0$, and then such a scaling is guaranteed to exist. However, the weights are not guaranteed to be positive and not even to be non–zero.

The hierarchical construction of a weighted tensor–product B–spline basis was discussed in [25]. Additional assumptions about the considered subdomains are required for defining a suitable normalized hierarchical basis of this kind, in particular to ensure positivity of the scaling factors w_γ. The truncated hierarchical basis — which will be studied in the next section — provides an alternative approach which does not require us to make any additional assumption on the considered subdomains.



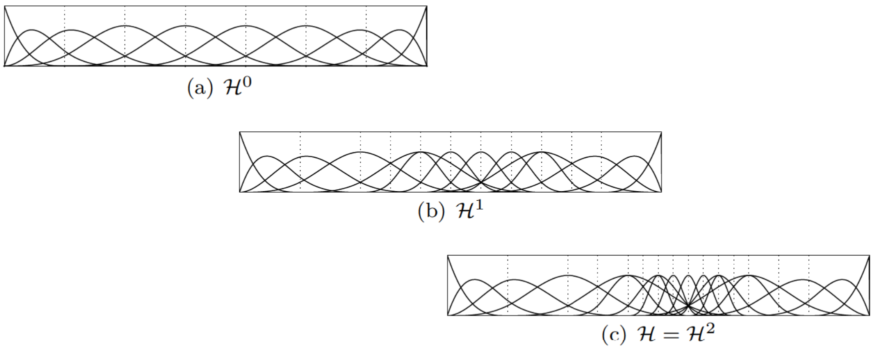
(a) knot vectors (b) subdomain hierarchy

(c) B–splines of level 2 (Γ^2) (d) hierarchical B–splines of level 2 ($\mathcal{H} \cap \Gamma^2$)

(e) B–splines of level 1 (Γ^1) (f) hierarchical B–splines of level 1 ($\mathcal{H} \cap \Gamma^1$)

(g) B–splines of level 0 (Γ^0) (h) hierarchical B–splines of level 0 ($\mathcal{H} \cap \Gamma^0$)

Fig. 3 Univariate hierarchical B–splines of degree 3 considered in Example 1 (part 2). The knot positions are visualized by vertical dotted lines.



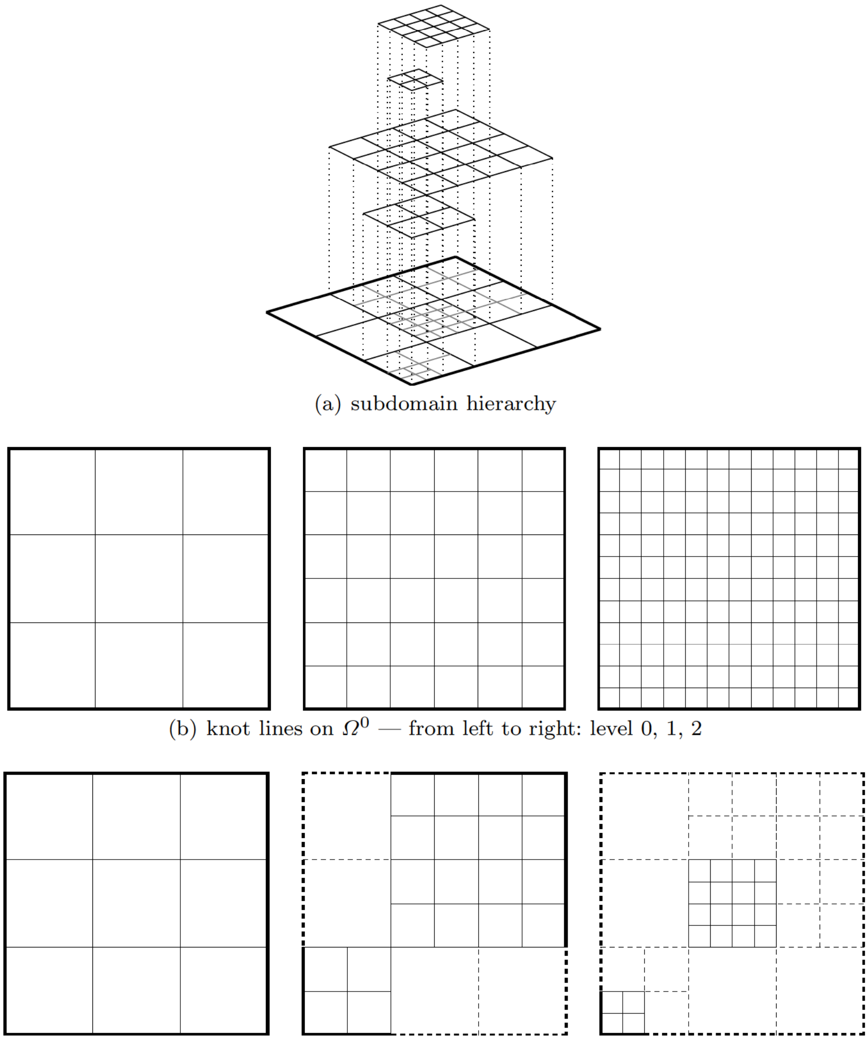
(a) $\mathcal{H}^0$

(b) $\mathcal{H}^1$

(c) $\mathcal{H} = \mathcal{H}^2$

Fig. 4 Hierarchical B–splines on 1, 2 and 3 levels defined over the subdomain hierarchy shown in Figure 3(b) and constructed according to Definition 2.

Example 1: univariate B–splines (part 2) An example of univariate hierarchical B–splines of degree 3 is illustrated in Figure 3. The 3–level hierarchy is defined starting by three nested knot vectors with knots of multiplicities 4 at the two extrema of the intervals and single knots elsewhere. In Figure 4 we show the hierarchical B–spline bases $\mathcal{H}^0$, $\mathcal{H}^1$ and $\mathcal{H}^2$ related to the subdomain hierarchy shown in Figure 3(b). In particular, $\mathcal{H}^0$ is built on Ω^0, $\mathcal{H}^1$ on $\Omega^0 \supseteq \Omega^1$, and $\mathcal{H}^2$ on $\Omega^0 \supseteq \Omega^1 \supseteq \Omega^2$. Note that $\mathcal{H} \cap \Gamma^\ell$, for some level ℓ, is not equal to $\mathcal{H}^\ell$, see Figure 3(h,f,d) versus Figure 4(a,b,c). The validity of properties (H1–H3) in Proposition 5 for this example can be verified through Figures 3 and 4.



(a) subdomain hierarchy

(b) knot lines on Ω^0 — from left to right: level 0, 1, 2

(c) subdomains — from left to right: Ω^0, Ω^1, Ω^2 (solid lines)

Fig. 5 Nested subdomains of Example 2 (part 2) with triple knots along the boundary of Ω^0 (bold lines) and single knots elsewhere.

Example 2: tensor–product B–splines (part 2) Figure 5 shows an example of a subdomain hierarchy with triple knots along the boundary of Ω^0 and single knots elsewhere. Triple knots are also taken along the parts of the boundary of Ω^1 and Ω^2 coinciding with the one of Ω^0. Some bi–quadratic hierarchical B–splines of level 0 and 1 defined on the considered hierarchical configuration are presented in Figures 6 and 7.

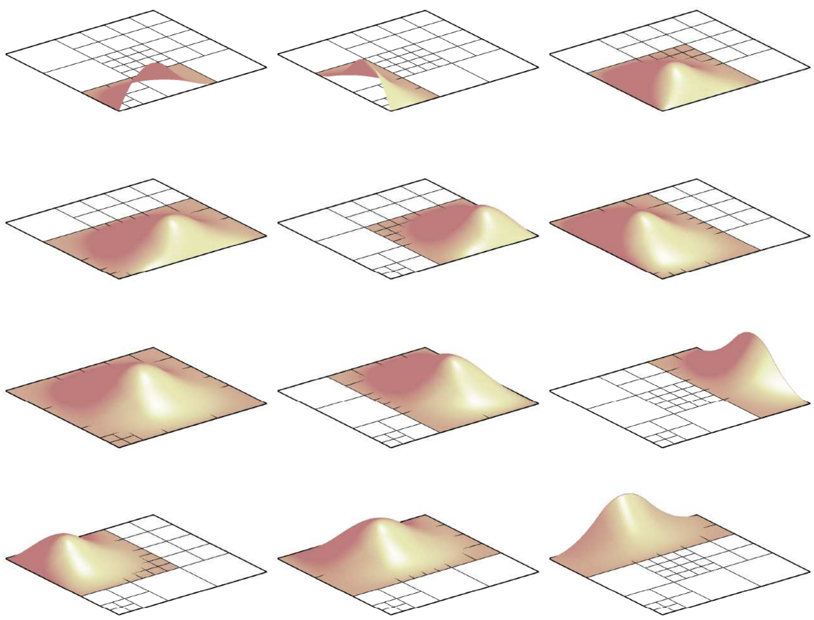


Fig. 6 Some bi–quadratic hierarchical B–splines of level 0 (in $\mathcal{H}\cap\Gamma^0$) considered in Example 2 (part 2).

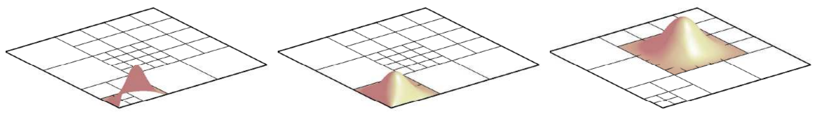


Fig. 7 Some bi–quadratic hierarchical B–splines of level 1 (in $\mathcal{H}\cap\Gamma^1$) considered in Example 2 (part 2).

4 The truncated hierarchical basis

We now propose a strategy to construct a normalized basis for the multilevel spline space S. The construction is based on the idea of eliminating the contributions of functions from finer levels in the hierarchy to coarser level ones. Let

$$f = \sum_{\gamma\in\Gamma^{\ell+1}} c_\gamma^{\ell+1}(f)\gamma \tag{7}$$

be a function $f \in V^\ell$ expressed in terms of the finer basis $\Gamma^{\ell+1}$, where $c_\gamma^k(f)$ denotes the coefficient of $f \in V^k$ with respect to the basis element $\gamma \in \Gamma^k$.

We decompose the sum in (7) into two parts by separating the contributions of the basis functions in $\Gamma^{\ell+1}$ whose support is or is not completely contained in the finer subdomain $\Omega^{\ell+1}$, namely

$$f = \sum_{\gamma \in \Gamma^{\ell+1},\, \mathrm{supp}^0 \gamma \subseteq \Omega^{\ell+1}} c_\gamma^{\ell+1}(f)\gamma \;+ \sum_{\gamma \in \Gamma^{\ell+1},\, \mathrm{supp}^0 \gamma \not\subseteq \Omega^{\ell+1}} c_\gamma^{\ell+1}(f)\gamma.$$

We define the *truncation* of f at level $\ell+1$ as the second sum on the right–hand side in the above equation, namely

$$\mathrm{trunc}^{\ell+1} f = \sum_{\gamma \in \Gamma^{\ell+1},\, \mathrm{supp}^0 \gamma \not\subseteq \Omega^{\ell+1}} c_\gamma^{\ell+1}(f)\gamma. \tag{8}$$

At each level ℓ, the truncation mechanism removes from the function f the contribution of basis functions in $\Gamma^{\ell+1}$ which are included in the hierarchical basis.

In combination with a selection procedure analogous to the one introduced by Definition 2, we can use the truncation operation to reduce the support of hierarchical basis functions of coarse levels. These observations lead us to the following definition of truncated hierarchical spline bases.

Definition 7 *Given a subdomain hierarchy* $(\Omega^\ell)_{\ell=0,\ldots,N-1}$*, the* truncated hierarchical basis $\mathcal{T}$ *is recursively generated by the following algorithm.*

(I) Initialization: $\mathcal{T}^0 = \mathcal{H}^0$.
(II) Recursive case: $\mathcal{T}^{\ell+1} = \mathcal{T}_A^{\ell+1} \cup \mathcal{T}_B^{\ell+1}$, *for* $\ell = 0, \ldots, N-2$, *where*

$$\mathcal{T}_A^{\ell+1} = \left\{ \mathrm{trunc}^{\ell+1}\tau : \tau \in \mathcal{T}^\ell \wedge \mathrm{supp}^0\, \tau \not\subseteq \Omega^{\ell+1} \right\} \quad \textit{and} \quad \mathcal{T}_B^{\ell+1} = \mathcal{H}_B^{\ell+1}.$$

(III) $\mathcal{T} = \mathcal{T}^{N-1}$.

The recursive construction of the truncated spline hierarchy can be phrased as follows.

- First, in view of the nested construction of the spaces, by expressing each basis function in $\mathcal{T}^\ell$ as a linear combination of the basis functions in $\Gamma^{\ell+1}$, we *truncate* the existing basis functions in $\mathcal{T}^\ell$ whose support is not contained in $\Omega^{\ell+1}$ (these are the basis functions collected in $\mathcal{T}_A^{\ell+1}$). Note that any basis function in $\mathcal{T}^\ell$ whose support is contained in $\Omega^{\ell+1}$ would be completely truncated, i.e., eliminated. These basis functions, however, are not considered since they are not included in the basis by the selection mechanism described in Definition 7.
- Second, the finer subdomain is covered by the functions in $\Gamma^{\ell+1}$ with support in $\Omega^{\ell+1}$ (these are the basis functions collected in $\mathcal{T}_B^{\ell+1}$).

The bases constructed in [10, 23] fit into this general framework of truncated hierarchical bases. In [23] it has been applied to Powell–Sabin B–splines on triangulations, while the truncated hierarchical B–spline basis (THB–spline basis) has been introduced in the recent paper [10] which considers the multivariate tensor–product setting.

Remark 8 For a practical construction of the truncated hierarchical basis, we do not need to consider full spline bases Γ^ℓ defined over the entire domain Ω, see Remark 3. Moreover, only a small number of basis functions will be truncated in each step of the definition, so we do not need to subdivide all coarser basis functions in terms of finer ones. This locality should be exploited in a practical implementation.

Let τ be a truncated hierarchical basis function and let γ be the hierarchical basis function in $\mathcal{H}^\ell$ of a certain level ℓ from which τ has been derived. Observe that either γ is introduced in the hierarchical basis from the very beginning, or it is selected at the recursive step of a certain level, i.e., $\gamma \in \mathcal{H}^0$ or $\gamma \in \mathcal{H}_B^{\ell+1}$, respectively. We say that γ is the *mother* of τ, and that τ is the *child* of γ, namely

$$\gamma = \mathrm{mot}(\tau) \;\Leftrightarrow\; \tau = \mathrm{trunc}^{N-1}\left(\mathrm{trunc}^{N-2}\left(\ldots\left(\mathrm{trunc}^{\ell+1}(\gamma)\right)\ldots\right)\right) \\ \Leftrightarrow\; \tau = \mathrm{child}(\gamma).$$

The *level* of τ is said to be the level of its mother, i.e., $\mathrm{lev}(\tau) = \ell$ if $\mathrm{mot}(\tau) \in \Gamma^\ell$. For any function $\gamma \in \Gamma^\ell$ without a child we define $\mathrm{child}(\gamma) = 0$.

Any function τ in $\mathcal{T}$ of level ℓ is derived from the corresponding mother γ by subtracting the contributions of basis functions defined on finer levels whose children are included in the truncated hierarchical basis. Consequently, the support of $\gamma - \tau$ is contained in $\Omega^{\ell+1}$, and the restrictions of τ and γ to $D^\ell = \Omega^\ell \setminus \Omega^{\ell+1}$ are identical,

$$\tau\big|_{D^\ell} = \mathrm{child}(\gamma)\big|_{D^\ell} = \gamma\big|_{D^\ell} = \mathrm{mot}(\tau)\big|_{D^\ell}. \tag{9}$$

The non–negativity and linear independence of the basis functions are preserved by the truncation mechanism, as well as the nested nature of the spline spaces defined level by level. In addition, the bases $\mathcal{H}$ and $\mathcal{T}$ span the same space S. We summarize these observations in the following proposition.

Proposition 9 *By assuming the properties (A1–A5) for the bases Γ^ℓ, the truncated hierarchical basis $\mathcal{T}$ possesses the following properties:*

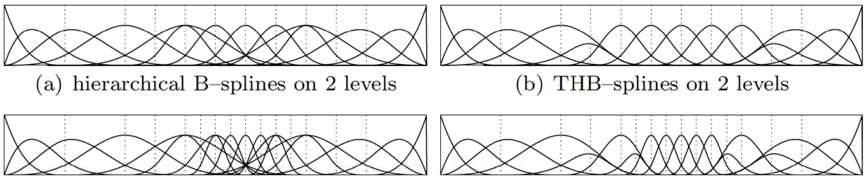
(T1) linear independence: $\sum_{\tau\in\mathcal{T}} c_\tau \tau = 0 \Leftrightarrow c_\tau = 0,\ \forall \tau \in \mathcal{T}$;
(T2) non–negativity: $\forall \tau \in \mathcal{T},\ \tau \geq 0$;
(T3) nested nature of the spline spaces: $\mathrm{span}\,\mathcal{T}^\ell \subseteq \mathrm{span}\,\mathcal{T}^{\ell+1}$, $\ell = 0,\ldots,N-2$;
(T4) the span of $\mathcal{T}$ is the multilevel spline space S which is determined by the subdomain hierarchy, as introduced in Definition 2, i.e. $S = \mathrm{span}\,\mathcal{H} = \mathrm{span}\,\mathcal{T}$.

Proof In order to prove (T1) we consider a linear combination of 0 in terms of truncated basis functions and decompose it into the contributions of basis functions of the same level,

$$\sum_{\tau\in\mathcal{T}} c_\tau\tau = \sum_{\tau\in\mathcal{T},\,\mathrm{lev}(\tau)=0} c_\tau\tau + \sum_{\tau\in\mathcal{T},\,\mathrm{lev}(\tau)=1} c_\tau\tau + \ldots + \sum_{\tau\in\mathcal{T},\,\mathrm{lev}(\tau)=N-1} c_\tau\tau = 0. \tag{10}$$

The basis functions collected by the first sum in (10) are the only non–zero functions acting on the region given by $D^0 = \Omega^0 \setminus \Omega^1$. In virtue of (9) and of the local linear independence of the basis Γ^0, these functions are locally linearly independent on D^0, and thus the corresponding coefficients c_τ must be zero. Excluding the functions already considered in this first sum, the basis functions collected by the second sum are the only non–zero functions which act on $D^1 = \Omega^1 \setminus \Omega^2$. As before, also the corresponding coefficients c_τ must be zero. Iterating this argument N times proves (T1).

Due to the two-scale relation (A5), the non–negativity of the basis functions in (T2) is also preserved by construction. Property (T3) and the equivalence of the two multilevel spline spaces (T4) can be shown by generalizing Lemma 8 and Theorem 9 in [10]. □



(a) hierarchical B–splines on 2 levels
(b) THB–splines on 2 levels
(c) hierarchical B–splines on 3 levels
(d) THB–splines on 3 levels

Fig. 8 Univariate hierarchical (left) and truncated hierarchical (right) B–splines of degree 3 considered in Example 1 (part 3) with respect to the same subdomain hierarchy as in Figure 3.

In view of property (T4) and Proposition 6, we may conclude that the space spanned by a truncated hierarchical basis defined over a sequence of subdomains $(\Omega^\ell)_{\ell=0,\dots,N-1}$ is contained in the span of a truncated hierarchical basis defined over another sequence $(\hat{\Omega}^\ell)_{\ell=0,\dots,\hat{N}-1}$, which is the nested enlargement of the initial sequence. In the remainder of the paper we study additional properties of truncated hierarchical bases.

Example 1: univariate B–splines (part 3) Figure 8 presents the truncation mechanism applied to the set of hierarchical B–splines depicted in Figure 3. We obtain the so–called univariate THB–splines, introduced in [10].

Example 2: tensor–product B–splines (part 3) Figures 9 and 10 show the truncated children of the bivariate tensor–product hierarchical B–splines considered in Figures 6 and 7, respectively. In [10] the use of the bivariate THB–spline model was compared in the context of data fitting with respect to the classical hierarchical B–spline construction described in Definition 2.

Remark 10 The truncation mechanism ensures that the corresponding basis functions have the same or smaller support than in the case of the classical hierarchical basis. Consequently, the overlap between truncated basis functions associated with different levels is reduced, and it leads to sparser matrices involved in related approximation problems, see [10]. By considering some restrictions on the configuration of the hierarchical sequence of subdomains, we can also derive bounds for the number of truncated basis functions acting on any point in the domain. This discussion is deferred to Appendix A.

5 The preservation of coefficients and its consequences

The truncated basis preserves the coefficients of a function which is represented with respect to one of the bases $\{\Gamma^\ell\}_{\ell=0,\dots,N-1}$. Lemma 11 and Theorem 12 describe this property.

Lemma 11 *Let $\tau\big|_{D^\ell}$ be the restriction of a truncated basis function $\tau \in \mathcal{T}$ to $D^\ell = \Omega^\ell \setminus \Omega^{\ell+1}$ with $\ell > \operatorname{lev}(\tau)$, and let*

$$\tau\big|_{D^\ell} = \sum_{\gamma \in \Gamma^\ell} c_\gamma^\ell(\tau)\gamma\big|_{D^\ell} \tag{11}$$

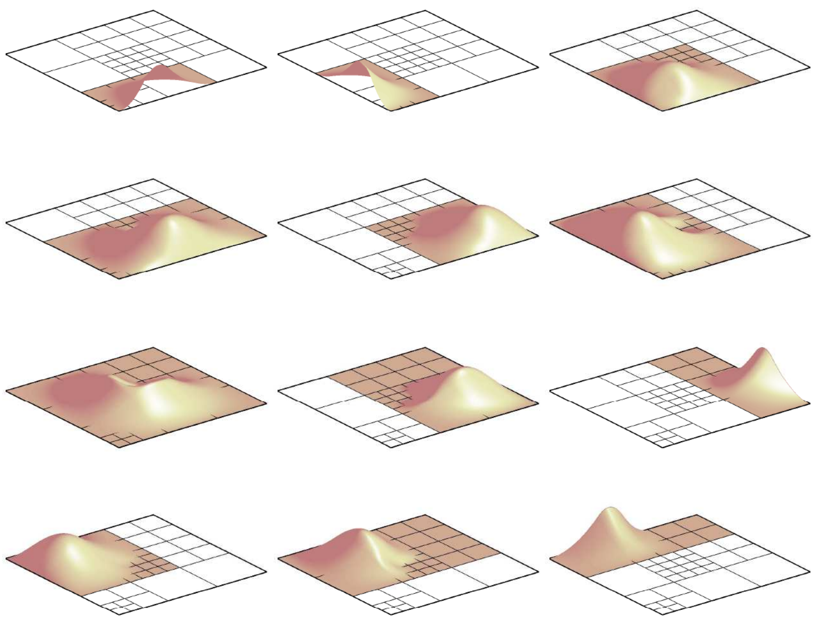


Fig. 9 Bi–quadratic THB–splines of level 0 considered in Example 2 (part 3) and related to the mother functions shown in Figure 6.

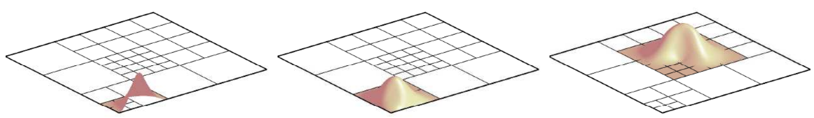


Fig. 10 Bi–quadratic THB–splines of level 1 considered in Example 2 (part 3) and related to the mother functions shown in Figure 7.

be its representation with respect to the basis Γ^ℓ. If $\gamma \in \Gamma^\ell$ possesses a child in $\mathcal{T}$ then $c^\ell_\gamma(\tau) = 0$.

Proof This is implied by the truncation operation defined in (8). Indeed, the truncation at level ℓ removes the contribution of any $\gamma \in \Gamma^\ell$ with a child in $\mathcal{T}$. □

In Theorem 12 we present the main result of this section. Together with the two consequent corollaries — Corollary 13 and 14 hereinafter — it allows us to extend the list of properties of the truncated construction. In addition, it plays a key role in the proof of Theorem 19 in Section 7, where the stability analysis of the basis is discussed.

Theorem 12 (Preservation of coefficients) *Let $f\big|_{D^\ell}$ be the restriction of a function $f \in S$ to $D^\ell = \Omega^\ell \setminus \Omega^{\ell+1}$. We consider its representations with respect to Γ^ℓ and $\mathcal{T}$,*

$$f\big|_{D^\ell} = \sum_{\gamma\in\Gamma^\ell} c_\gamma^\ell(f)\gamma\big|_{D^\ell} = \sum_{\tau\in\mathcal{T}} d_\tau(f)\tau\big|_{D^\ell}. \tag{12}$$

For each $\tau \in \mathcal{T}$ with $\operatorname{lev}(\tau) = \ell$ there exists $\gamma \in \Gamma^\ell$ such that $\gamma = \operatorname{mot}(\tau)$ and the coefficients of τ and γ in the two representations (12) *are identical,*

$$d_\tau(f) = c_\gamma^\ell(f). \tag{13}$$

Proof Since $f \in S$ we have $f\big|_{D^\ell} \in V^\ell\big|_{D^\ell}$. Thus, we can represent $f\big|_{D^\ell}$ with respect to both Γ^ℓ and $\mathcal{T}$, in terms of basis functions whose supports overlap D^ℓ as in (12). The rightmost sum in (12) can be decomposed into

$$\sum_{\tau\in\mathcal{T}} d_\tau(f)\tau\big|_{D^\ell} = \underbrace{\sum_{\tau\in\mathcal{T},\,\operatorname{lev}(\tau)>\ell} d_\tau(f)\tau\big|_{D^\ell}}_{\text{(i)}} + \underbrace{\sum_{\tau\in\mathcal{T},\,\operatorname{lev}(\tau)=\ell} d_\tau(f)\tau\big|_{D^\ell}}_{\text{(ii)}} + \underbrace{\sum_{\tau\in\mathcal{T},\,\operatorname{lev}(\tau)<\ell} d_\tau(f)\tau\big|_{D^\ell}}_{\text{(iii)}}.$$

The three sums (i–iii) are now analyzed separately.

(i) This sum is empty. Indeed, the definition of the truncated basis implies that $\operatorname{supp}^0(\tau) \subseteq \Omega^{\operatorname{lev}(\tau)}$ and $\Omega^{\operatorname{lev}(\tau)} \subseteq \Omega^{\ell+1}$, consequently $\operatorname{supp}^0(\tau) \cap D^\ell = \emptyset$.

(ii) For any $\tau \in \mathcal{T}$ with $\operatorname{lev}(\tau) = \ell$, there exists $\gamma = \operatorname{mot}(\tau) \in \Gamma^\ell$ so that the restrictions of γ and τ to D^ℓ coincide, see (9). Consequently, the second sum can be rewritten as

$$\sum_{\tau\in\mathcal{T},\,\operatorname{lev}(\tau)=\ell} d_\tau(f)\tau\big|_{D^\ell} = \sum_{\gamma\in\Gamma^\ell,\,\operatorname{child}(\gamma)\in\mathcal{T}} d_{\operatorname{child}(\gamma)}(f)\gamma\big|_{D^\ell}. \tag{14}$$

(iii) In view of the truncation process, we will prove that truncated basis functions introduced at levels less than ℓ in the hierarchy can only contribute in terms of basis functions of Γ^ℓ that have no child in $\mathcal{T}$. Indeed, we can rewrite the corresponding truncated basis functions τ in terms of the basis Γ^ℓ,

$$\sum_{\tau\in\mathcal{T},\,\operatorname{lev}(\tau)<\ell} d_\tau(f)\tau\big|_{D^\ell} = \sum_{\tau\in\mathcal{T},\,\operatorname{lev}(\tau)<\ell} d_\tau(f)\left(\sum_{\gamma\in\Gamma^\ell,\,\operatorname{child}(\gamma)\in\mathcal{T}} c_\gamma^\ell(\tau)\gamma\big|_{D^\ell} + \sum_{\gamma\in\Gamma^\ell,\,\operatorname{child}(\gamma)\notin\mathcal{T}} c_\gamma^\ell(\tau)\gamma\big|_{D^\ell}\right). \tag{15}$$

According to Lemma 11 the coefficients $c_\gamma^\ell(\tau)$ for all $\gamma \in \Gamma^\ell$ with $\operatorname{child}(\gamma) \in \mathcal{T}$ vanish, hence we may omit the first term in the brackets. After swapping the order of summations we arrive at

$$\sum_{\tau\in\mathcal{T},\,\operatorname{lev}(\tau)<\ell} d_\tau(f)\tau\big|_{D^\ell} = \sum_{\gamma\in\Gamma^\ell,\,\operatorname{child}(\gamma)\notin\mathcal{T}} \underbrace{\left(\sum_{\tau\in\mathcal{T},\,\operatorname{lev}(\tau)<\ell} d_\tau(f)c_\gamma^\ell(\tau)\right)}_{=d_\gamma^\ell(f)} \gamma\big|_{D^\ell}. \tag{16}$$

In order to complete the proof we use the results from (i–iii), in particular the relations in equations (14)–(16), and compare with the representation of $f\big|_{D^\ell}$ in (12),

$$\begin{aligned} f\big|_{D^\ell} &= \sum_{\gamma\in\Gamma^\ell} c_\gamma^\ell(f)\gamma\big|_{D^\ell} = \sum_{\tau\in\mathcal{T}} d_\tau(f)\tau\big|_{D^\ell} \\ &= \sum_{\gamma\in\Gamma^\ell,\,\mathrm{child}(\gamma)\notin\mathcal{T}} d_\gamma^\ell(f)\gamma\big|_{D^\ell} + \sum_{\gamma\in\Gamma^\ell,\,\mathrm{child}(\gamma)\in\mathcal{T}} d_{\mathrm{child}(\gamma)}(f)\gamma\big|_{D^\ell}, \end{aligned}$$

where $d_\gamma^\ell(f)$ is defined in (16). Comparing the coefficient with respect to any $\gamma\in\Gamma^\ell$ satisfying $\gamma=\mathrm{mot}(\tau)$ for some $\tau\in\mathcal{T}$, which is equivalent to $\tau=\mathrm{child}(\gamma)\in\mathcal{T}$, implies (13). □

This theorem implies that truncated hierarchical basis functions form a partition of unity, and that they inherit the Greville points from their mothers. This is detailed in the next two corollaries.

Corollary 13 *By assuming the properties (A1–A5) for the bases Γ^ℓ, the truncated hierarchical basis $\mathcal{T}$ possesses the following property:*

(T5) the truncated hierarchical basis $\mathcal{T}$ forms a convex partition of unity on Ω^0.

Proof By assumption (A4), we can represent the function $f=1$ on Ω^0 as the sum of all basis functions in Γ^ℓ,

$$1=\sum_{\gamma\in\Gamma^\ell} c_\gamma^\ell(1)\gamma, \qquad c_\gamma^\ell(1)=1, \tag{17}$$

for each $\ell=0,\dots,N-1$. Since $V^0\big|_{\Omega^0}\subseteq\operatorname{span}\mathcal{T}\big|_{\Omega^0}$, this function on Ω^0 can also be represented in terms of the truncated basis $\mathcal{T}$,

$$1=\sum_{\tau\in\mathcal{T}} d_\tau(1)\tau, \tag{18}$$

with certain coefficients $d_\tau(1)$. Consider an arbitrary but fixed $\tau\in\mathcal{T}$ and let $\ell=\mathrm{lev}(\tau)$. The two representations (17) and (18) are also valid when restricted to

$$D^\ell=\Omega^\ell\setminus\Omega^{\ell+1}\subseteq\Omega^0.$$

Since τ has a mother $\gamma=\mathrm{mot}(\tau)\in\Gamma^\ell$, Theorem 12 implies that

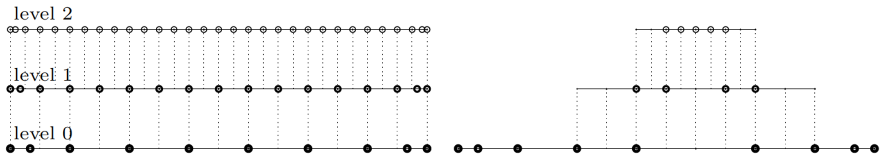
$$d_\tau(1)=c_\gamma^\ell(1)=1.$$

Finally, the non–negativity (see property (T2) of Proposition 9) gives the convex partition of unity. □

Let $\mathbb{P}_1$ be the space of n–variate linear polynomials, and let x_k, $k=1,\dots,n$, be the linear monomials of $\mathbb{P}_1$. Assuming that $\mathbb{P}_1$ is a subspace of the spline space V^ℓ, then the corresponding *Greville points* $\mathbf{g}_\gamma^\ell=(g_{\gamma,1}^\ell,\dots,g_{\gamma,n}^\ell)$ are defined as the coefficients in the spline representations of the linear monomials [11,20], i.e.,

$$x_k=\sum_{\gamma\in\Gamma^\ell} g_{\gamma,k}^\ell\gamma, \qquad k=1,\dots,n, \tag{19}$$

or, equivalently, $g_{\gamma,k}^\ell=c_\gamma^\ell(x_k)$. It is then possible to define control polygons/control nets for spline functions, by considering coefficients in $\mathbb{R}^{n+1}$ defined in terms of the Greville points.



(a) Greville points of underlying B–splines (b) Greville points of THB–splines

Fig. 11 Greville points of THB–splines of degree 3 defined over the subdomain hierarchy shown in Figure 3, see Example 1 (part 4).

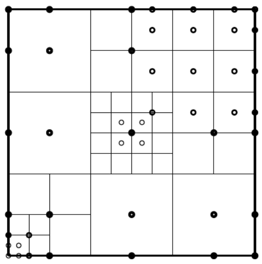


Fig. 12 Greville points of bi–quadratic THB–splines defined over the subdomain hierarchy shown in Figure 5, see Example 2 (part 4).

Corollary 14 *If the coarsest space V^0 contains the space of linear polynomials $\mathbb{P}_1$, then Greville points can be associated with each basis Γ^ℓ and with the truncated hierarchical basis. We have the following simple relation between the Greville points:*

(T6) the Greville point of each basis function in $\mathcal{T}$ coincides with the one of its mother,

$$\mathbf{g}_\tau = \mathbf{g}^{\mathrm{lev}(\tau)}_{\mathrm{mot}(\tau)}.$$

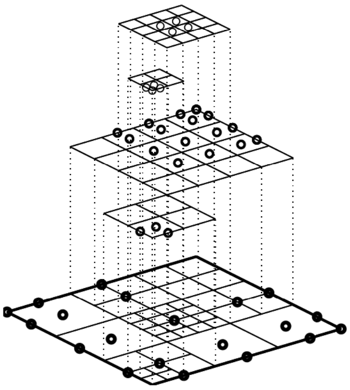
Proof Since $V^0\big|_{\Omega^0} \subseteq \operatorname{span} \mathcal{T}\big|_{\Omega^0}$, the considered polynomials in (19) can be represented in terms of the truncated basis $\mathcal{T}$. In virtue of Theorem 12, we can proceed as in the proof of Corollary 13 to show that the original coefficients in (19) are preserved by the truncation mechanism. □

Example 1: univariate B–splines (part 4) The Greville points of the THB–splines of degree 3 reported in Figure 8(d) are shown in Figure 11.

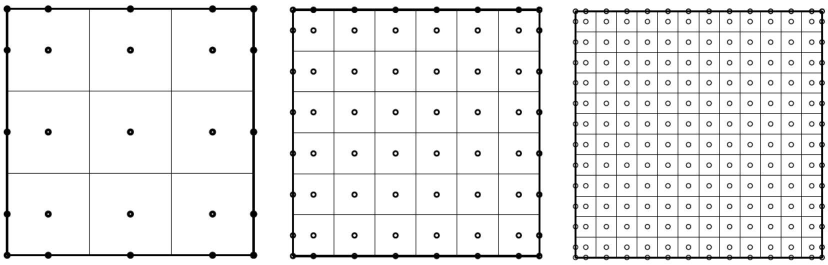
Example 2: tensor–product B–splines (part 4) The Greville points of the bi–quadratic THB–splines defined over the subdomain hierarchy introduced in Figure 5 are shown in Figure 12. Figure 13 relates the Greville points of truncated hierarchical B–splines to the ones of their mothers.

6 Locally stable bases

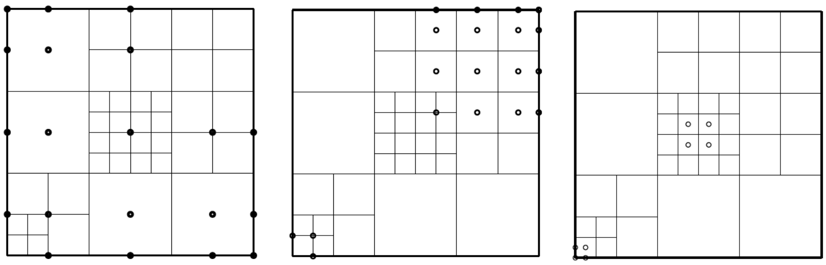
Before analyzing the stability of a hierarchical basis, we first introduce the concept of a locally stable basis.



(a) level 0, 1, 2



(b) Greville points of underlying B–splines — from left to right: level 0, 1, 2



(c) Greville points of THB–splines — from left to right: level 0, 1, 2

Fig. 13 Greville points of bi–quadratic THB–splines defined over the subdomain hierarchy shown in Figure 5, see Example 2 (part 4).

Definition 15 *Consider an n–variate spline function*

$$f(\mathbf{u}) = \sum_{\gamma \in \Gamma} c_\gamma(f)\gamma(\mathbf{u}), \qquad \mathbf{u} = (u_1, \ldots, u_n) \in \Omega,$$

which is represented in a certain basis Γ satisfying (A1–A2). We denote by C_π a vector of coefficients c_γ corresponding to all basis functions $\gamma \in \Gamma$ satisfying $\operatorname{supp} \gamma \supseteq \pi$. *The basis Γ is said to be* locally stable *if there exist two constants k_0 and k_1 such that the inequalities*

$$k_0 \|C_\pi\| \leq \|f\|_\pi \leq k_1 \|C_\pi\| \tag{20}$$

are satisfied for any vector of coefficients C_π and for any cell $\pi \in \Pi$ related to the basis Γ.

Note that (20) involves two norms: one for functions (applied to f) and one for vectors (applied to C_π). For simplicity we use the same symbol for both norms, which should be of the same type. In particular, we consider the L_∞ (supremum) norm for functions and the maximum norm for vectors,

$$\|C\| = \|C\|_\infty = \max_{c \in C} |c|, \quad \text{and} \quad \|f\|_\pi = \|f\|_{\infty,\pi} = \max_{\mathbf{u} \in \pi} |f(\mathbf{u})|.$$

Besides (A1–A5) introduced in Section 2, we introduce an additional assumption on the bases Γ^ℓ used in the hierarchical setting:

(A6) the bases Γ^ℓ are assumed to be locally stable on the cells $\pi \in \Pi^\ell$, $\ell = 0, \ldots, N-1$, in terms of stability constants which are independent of the level ℓ.

We will use this assumption later on to prove strong stability of the truncated basis. More precisely, there exists a constant K such that

$$|c_\gamma^\ell| \leq K \| \sum_{\gamma \in \Gamma^\ell} c_\gamma^\ell \gamma \|_{\infty,\pi} \tag{21}$$

holds for all levels ℓ, for all cells $\pi \in \Pi^\ell$ and for any basis function γ in Γ^ℓ satisfying $\operatorname{supp}^0 \gamma \supseteq \pi$. Note that the upper bound

$$\| \sum_{\gamma \in \Gamma^\ell} c_\gamma^\ell \gamma \|_{\infty,\pi} \leq \max\{|c_\gamma^\ell| : \operatorname{supp}^0 \gamma \supseteq \pi\}$$

follows directly from the convex partition of unity (A3–A4) of the basis Γ^ℓ.

Example 1: univariate B–splines (part 5) Unlike properties (A1–A5), property (A6) — local stability — for the univariate B–spline case needs to be discussed in more detail. This discussion is deferred to Appendix B. We present here the related result. If there exists a positive constant T so that

$$\frac{t_{r+d}^\ell - t_{r-d+1}^\ell}{t_{r+1}^\ell - t_r^\ell} \leq T, \tag{22}$$

holds for all non–empty knot spans $]t_r^\ell, t_{r+1}^\ell[\in \Pi^\ell$ and for $\ell = 0, \ldots, N-1$, then all B–spline bases Γ^ℓ of degree d satisfy assumption (A6). Inequality (22) is trivially satisfied when using uniform knot sequences, see for instance Figure 3.

Remark 16 As mentioned in Remark 4, B–spline bases Γ^ℓ with different degrees d^ℓ could be chosen at different levels ℓ in the hierarchical framework. However, we need to choose all these degrees carefully, in order to ensure that there exists a constant K in (21) which does not depend on the number of levels. This requirement is satisfied when all the degrees can be bounded from above by a constant independent of the number of levels.

We now consider the tensor–product of two spline spaces in a hierarchical construction, and we will show that the properties (A1–A6) are preserved for such tensor–product spaces.

Lemma 17 *Consider two hierarchies of nested spline spaces V^ℓ and $\hat{V}^\ell$, $\ell = 0, \ldots, N-1$, with domains $\Omega \subset \mathbb{R}^n$ and $\hat{\Omega} \subset \mathbb{R}^{\hat{n}}$. If both hierarchies of spaces satisfy the assumptions* (A1–A6)*, then so does the hierarchy of tensor–product spaces $V^\ell \otimes \hat{V}^\ell$ with the domain $\Omega \times \hat{\Omega}$.*

Proof The proof of (A1–A5) is obvious. In order to prove (A6), we consider the restriction of a tensor–product function f to a cell $\pi \times \hat{\pi}$, where $\pi \in \Pi^\ell$ and $\hat{\pi} \in \hat{\Pi}^\ell$,

$$f\big|_{\pi\times\hat{\pi}} = \sum_{\gamma\in\Gamma^\ell} \sum_{\hat{\gamma}\in\hat{\Gamma}^\ell} c_{\gamma\hat{\gamma}}\, \gamma\big|_\pi\, \hat{\gamma}\big|_{\hat{\pi}}\,. \tag{23}$$

Since Γ^ℓ satisfies (A6), there exists a constant K such that

$$|c_{\gamma\hat{\gamma}}| \le K\| \sum_{\gamma\in\Gamma^\ell} c_{\gamma\hat{\gamma}}\, \gamma\|_{\infty,\pi} = K\|\,|\sum_{\gamma\in\Gamma^\ell} c_{\gamma\hat{\gamma}}\, \gamma|\,\|_{\infty,\pi} \tag{24}$$

holds for each γ satisfying $\operatorname{supp}\gamma \supseteq \pi$. Similarly, since $\hat{\Gamma}^\ell$ satisfies (A6), there exists a constant $\hat{K}$ such that

$$|\sum_{\gamma\in\Gamma^\ell} c_{\gamma\hat{\gamma}}\, \gamma(\mathbf{u})| \le \hat{K}\| \sum_{\hat{\gamma}\in\hat{\Gamma}^\ell} \left(\sum_{\gamma\in\Gamma^\ell} c_{\gamma\hat{\gamma}}\, \gamma(\mathbf{u}) \right) \hat{\gamma}\|_{\infty,\hat{\pi}} \tag{25}$$

holds for each $\hat{\gamma}$ satsfying $\operatorname{supp}\hat{\gamma} \supseteq \hat{\pi}$ and for all $\mathbf{u} \in \pi$. Combining these two bounds gives

$$\begin{aligned} |c_{\gamma\hat{\gamma}}| &\le K\|\,|\sum_{\gamma\in\Gamma^\ell} c_{\gamma\hat{\gamma}}\, \gamma\,|\,\|_{\infty,\pi} \le K\hat{K}\|\|\sum_{\gamma\in\Gamma^\ell}\sum_{\hat{\gamma}\in\hat{\Gamma}^\ell} c_{\gamma\hat{\gamma}}\, \gamma\hat{\gamma}\|_{\infty,\hat{\pi}}\|_{\infty,\pi} \\ &\le K\hat{K}\| \sum_{\gamma\in\Gamma^\ell}\sum_{\hat{\gamma}\in\hat{\Gamma}^\ell} c_{\gamma\hat{\gamma}}\gamma\hat{\gamma}\|_{\infty,\pi\times\hat{\pi}}, \end{aligned}$$

for all products $\gamma\hat{\gamma}$ satisfying $\operatorname{supp}\gamma\hat{\gamma} \supseteq \pi \times \hat{\pi}$. □

Example 2: tensor–product B–splines (part 5) Consider a set of univariate B–spline spaces satisfying (A1–A6). According to Lemma 17, these properties are inherited by their tensor–product spaces.

7 Strong stability of truncated hierarchical bases

We will now extend the notions of *weakly* and *strongly* stable wavelet bases, which were introduced in the context of multiresolution analysis (see, e.g., [5]), to our setting. Note that the multilevel spline spaces depend on the number of levels, but also on the choice of the subdomain hierarchy. This is different from the case of stability of wavelet expansions, which depend solely on the number of levels.

Consequently, weak and strong stability is a property which is shared by the bases of all multilevel spline spaces that can be generated from a given sequence of spaces V^ℓ with domain Ω by specifying different subdomain hierarchies. This means that the weak and strong stability is not just a property of a specific basis, but it has to be considered as a property of a construction which generates a basis for the adaptively refined multilevel spline space S from a given subdomain hierarchy.

Definition 18 *Consider an n–variate spline function*

$$f(\mathbf{u}) = \sum_{\gamma \in \mathcal{B}} c_\gamma(f)\gamma(\mathbf{u}), \qquad \mathbf{u} = (u_1, \ldots, u_n) \in \Omega^0,$$

which is represented in a certain basis $\mathcal{B} \in \{\mathcal{H}, \mathcal{T}, \ldots\}$ of the multilevel space $S = \operatorname{span} \mathcal{H} = \operatorname{span} \mathcal{T}$.

A construction of a basis for the multilevel spline space S is said to be strongly stable *if there exist two constants k_0 and k_1 such that the inequalities*

$$k_0 \|C\| \leq \|f\|_{\Omega^0} \leq k_1 \|C\| \tag{26}$$

are satisfied for any vector of coefficients $C = (c_\gamma)_{\gamma \in \mathcal{B}}$ related to the hierarchical basis $\mathcal{B}$ and for any subdomain hierarchy $(\Omega^\ell)_{\ell=0,\ldots,N-1}$.

It is said to be weakly stable *if there exist two polynomials $k_0(N)$ and $k_1(N)$ such that the inequalities* (26) *are satisfied for any vector of coefficients $C = (c_\gamma)_{\gamma \in \mathcal{B}}$ related to the hierarchical basis $\mathcal{B}$ defined on any subdomain hierarchy $(\Omega^\ell)_{\ell=0,1,\ldots,N-1}$, where N is the depth of the subdomain hierarchy.*

In other words, the two constants (or polynomials) k_0 and k_1, which identify strong (or weak) stability in the above definition, are not just related to a particular basis, but to all the hierarchical bases that can be obtained according to the given construction by specifying different possible subdomain hierarchies.

Note again that (26) involves two norms. In particular, we consider the L_∞ norm for functions and the maximum norm for vectors.

In the case of B–spline spaces, Kraft [15] showed that the construction of $\mathcal{H}$ described in Definition 2 is weakly stable, provided that the nested subdomains Ω^ℓ satisfy certain conditions. In addition, he provided a bi–quadratic counterexample with respect to dyadic refinement (see Figure 15(a)) for strong stability of $\mathcal{H}$ showing that k_1 in (26) has to depend at least linearly on the number of hierarchical levels. Appendix B presents an example which generalizes this argument to multivariate tensor–product B–splines of any degree.

For analyzing the stability properties of the *truncated* hierarchical basis, we need a mild assumption regarding the domain configuration:

(D) the closure of the subdomain $\Omega^{\ell+1} \subseteq \Omega^\ell$ coincides with the closure of a collection of cells $\pi \in \Pi^\ell$ of level ℓ for $\ell = 0, \ldots, N-1$, and the closure of Ω^0 coincides with the closure of a collection of cells $\pi \in \Pi^0$.

Note that, for hierarchical tensor–product B–splines, this condition on the domain configuration is much milder than the domain restrictions considered in [15].

By using again the preservation of coefficients, we are able to prove strong stability of truncated hierarchical bases. In the specific case of uniform hierarchical Powell–Sabin B–splines, it has been shown already that the truncation procedure leads to a strongly stable basis with respect to the supremum norm, see [23].

Theorem 19 *By assuming the properties (A1–A6) and (D) for the bases Γ^ℓ, the truncated hierarchical basis $\mathcal{T}$ possesses the following property:*

(T7) the construction of the truncated hierarchical basis is strongly stable with respect to the supremum norm.

Proof The stability estimate (26) needed to prove stability of the truncated basis $\mathcal{T}$ with respect to the considered norms can be rewritten as

$$k_0 \max_{\tau\in\mathcal{T}} |d_\tau| \leq \max_{\mathbf{u}\in\Omega^0} |f(\mathbf{u})| \leq k_1 \max_{\tau\in\mathcal{T}} |d_\tau| \quad (27)$$

where $d_\tau = d_\tau(f)$ is the coefficient of f with respect to $\tau \in \mathcal{T}$. We have to prove that there exist constants k_0 and k_1 that are independent of the number of levels N. Since the truncated hierarchical basis satisfies the convex partition of unity property, the inequality on the right in (27) holds with $k_1 = 1$. The proof of the inequality on the left in (27) is more subtle.

Consider an arbitrary but fixed basis function $\tau^\star \in \mathcal{T}$ introduced at level $\ell = \text{lev}(\tau^\star)$ and truncated, if needed, at subsequent levels. Since $\tau^\star$ belongs to the truncated hierarchical basis, and due to assumption (D), at least one cell $\pi \in \Pi^\ell$ satisfying $\pi \subseteq \Omega^\ell \setminus \Omega^{\ell+1}$ is necessarily part of the support of $\tau^\star$. The restriction of f to this cell, indicated as $f\big|_\pi$, can be represented both in terms of basis functions of Γ^ℓ and of the truncated hierarchical basis,

$$f\big|_\pi = \sum_{\gamma\in\Gamma^\ell} c_\gamma^\ell(f)\gamma\big|_\pi = \sum_{\tau\in\mathcal{T}} d_\tau(f)\tau\big|_\pi .$$

By Theorem 12 we know that $d_{\tau^\star}(f) = c^\ell_{\text{mot}(\tau^\star)}(f)$. In addition, assumption (A6) implies

$$|d_{\tau^\star}(f)| = |c^\ell_{\text{mot}(\tau^\star)}| \leq K\|f\|_{\infty,\pi} \leq K\|f\|_{\infty,\Omega^0}.$$

Applying this to all $\tau \in \mathcal{T}$ and choosing $k_0 = 1/K$ completes the proof of (27). □

Example 1: univariate B–splines (part 6) In virtue of the analysis in the previous section, when the knot vectors satisfy (22), for $\ell = 0, \ldots, N-1$, univariate truncated hierarchical B–spline bases obtained for any choice of subdomains $\Omega^0, \ldots, \Omega^{N-1}$ respecting (D) satisfy the hypothesis of Theorem 19. Consequently, they are strongly stable with respect to the L_∞ norm.

Example 2: tensor–product B–splines (part 6) In virtue of Lemma 17, the stability result related to the univariate B–spline case can be generalized to the multivariate tensor–product setting as follows. Consider a set of univariate B–spline spaces satisfying (A1–A6). According to Lemma 17, these properties are inherited by their tensor–product spaces. Strong stability of the THB–splines is then implied by Theorem 19. Hence, the truncated hierarchical bases for sequences of nested multivariate tensor–product spline spaces of multi–degree $\mathbf{d} = (d_1, \ldots, d_n)$ with associated nested subdomains $\Omega^\ell \subset \mathbb{R}^n$ are strongly stable, provided that the subdomain hierarchies satisfy assumption (D) and the knot vectors satisfy (22).

8 Closure

The construction and analysis of suitable hierarchical bases has been presented for a class of multilevel spline spaces, which enable a local mesh refinement in an effective and easy way. It has been shown that truncated hierarchical bases have many interesting properties, see properties (T1–T7). In particular, truncated splines are non–negative, they span nested spaces, they preserve the coefficients of a function represented in terms of the underlying basis (in particular the Greville points), they form a convex partition of unity, and they are strongly stable with respect to the supremum norm. Consequently, the truncation framework seems to be a promising adaptive approximation tool, in terms of its relatively simple construction and the properties of the spline hierarchy it defines.

The presented hierarchical framework can be applied to a wide range of spline spaces, as long as they possess the properties (A1–A5). We have illustrated the truncation procedure and its properties in the context of univariate and multivariate tensor–product B–splines. Other spaces of interest that satisfy these properties are NURBS [19], many generalized B–splines [4], type–I box splines [16], and Powell–Sabin B–splines with triadic refinement [23].

A Restricted hierarchies

The purpose of this section is to identify the advantage of the smaller supports which characterize the truncated basis compared to the classical hierarchical basis. The enhanced locality in the truncation framework could be suitably exploited in many applications. For example, the number of basis functions which have some influence on a single mesh element has to be taken into account in the characterization of the matrices involved in related approximation problems. For truncated hierarchical bases there exist domain configurations for which it is possible to bound the number of basis functions acting on any given point. In particular, by considering a specific class of restricted hierarchies, we prove that only truncated basis functions of at most two levels are different from zero on each point in the domain. On the other hand, we also provide a counterexample to show that this property is not true for classical hierarchical bases, which may have all levels involved also for this kind of domain configurations.

Similarly to the definition in Kraft's thesis [15] with respect to the B–spline case, we now additionally consider the auxiliary subdomains

$$\omega^\ell = \{\mathbf{p} \in \Omega^\ell \,|\, \forall \gamma \in \Gamma^\ell : \mathbf{p} \in \operatorname{supp}\gamma \Rightarrow \operatorname{supp}\gamma \subseteq \Omega^\ell\},$$

for $\ell = 0, \ldots, N-1$. By taking into account the effect of the truncation mechanism introduced in Section 4, we can prove the following result.

Proposition 20 *If $\Omega^{\ell+1} \subseteq \omega^\ell$, for $\ell = 0, \ldots, N-2$, then for any point $\mathbf{p}$ in Ω^0 the elements in the truncated basis which take non–zero values at $\mathbf{p}$ belong to at most two different levels.*

Proof We first note that any point $\mathbf{p}$ belongs to Ω^{N-1} or $D^\ell = \Omega^\ell \setminus \Omega^{\ell+1}$ for a certain value of the level ℓ, and that ω^ℓ represents the biggest subset of Ω^ℓ such that $\mathcal{H}_B^\ell$ spans the restriction of V^ℓ to ω^ℓ. For any $\tau \in \mathcal{T}$ with $\operatorname{lev}(\tau) = \ell - 1$, the function $\operatorname{trunc}^\ell \tau$ defined by (8) is a linear combination of basis functions $\gamma \in \Gamma^\ell$ so that $\operatorname{supp}^0 \gamma \cap \omega^\ell = \emptyset$. Hence, any truncated basis function τ of level $\ell - 1$ has its support entirely on $\Omega^{\ell-1} \setminus \omega^\ell$, and

$$\operatorname{supp}^0(\operatorname{trunc}^\ell \tau) \cap \omega^\ell = \emptyset.$$

If $\Omega^{\ell+1} \subseteq \omega^\ell$, for $\ell = 1, \ldots, N-2$, then also $\operatorname{supp}^0(\operatorname{trunc}^\ell \tau) \cap \Omega^{\ell+1} = \emptyset$. This means that the truncated child of a basis function introduced at level $\ell - 1$ will be non–zero on $\Omega^{\ell-1} \setminus \Omega^{\ell+1}$. In other words, only the supports of truncated basis functions of level $\ell - 1$ and ℓ will contain cells that belong to $D^\ell = \Omega^\ell \setminus \Omega^{\ell+1}$, for $\ell = 1, \ldots, N-2$. Similarly, only the supports of truncated basis functions of level $N-2$ and $N-1$ contain cells that belong to $\Omega^{N-1} \setminus \omega^{N-1}$, and only truncated basis functions of level $N-1$ contain cells inside ω^{N-1}. □

If the hierarchical configuration of the subdomains satisfies the hypothesis of Proposition 20, then the number of truncated basis functions acting on any given point of the domain is bounded.

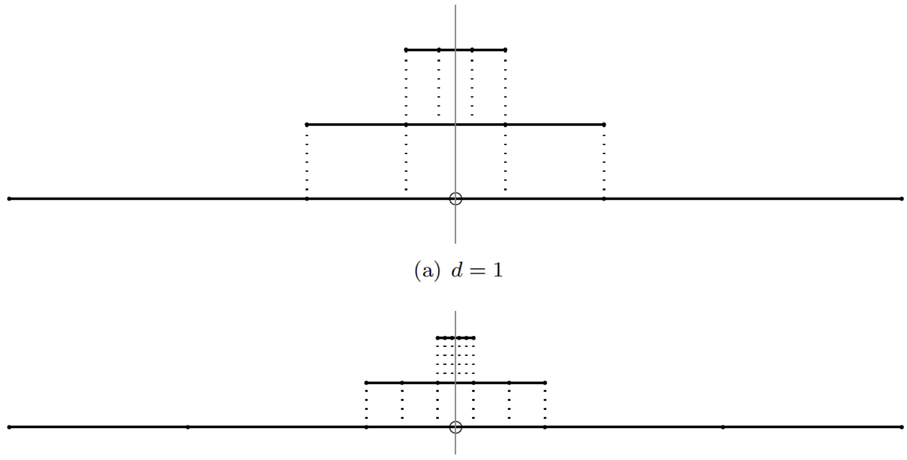


Fig. 14 Counterexample for Proposition 20 in the case of hierarchical B–splines in Remark 21.

Example 1 and 2: univariate and tensor–product B–splines (part 7) For univariate and multivariate tensor–product B–splines of degree d/multi–degree $\mathbf{d} = (d_1, \ldots, d_n)$, when the hypothesis of Proposition 20 is satisfied, the number of truncated basis functions which are non–zero on each cell of any level is then at most

$$2(d+1) \quad \text{and} \quad 2\prod_{i=1}^{n}(d_i+1),$$

respectively. These values do not depend on the number of levels.

Remark 21 *The result presented in Proposition 20 does not hold for hierarchical B–spline bases constructed according to Definition 2. Let us consider for example the following univariate configuration:*

- *an initial interval composed of $2d+2$ uniformly spaced single knots ($2d+1$ knot spans);*
- *at each refinement step only the central subinterval is subdivided in $2d+1$ uniform parts.*

Figure 14 shows the case $d = 1, 2$ after two refinement steps. At the midpoint of the initial interval all the hierarchical B–splines from level 1 to $N-1$ are non–zero.
On the other hand, the hypothesis in Proposition 20 is satisfied for this mesh configuration, so only THB–basis functions from at most two different levels will be non–zero at each point.

B Proof of (A6) for univariate B–splines

Lemma 22 *We assume that there exists a positive constant T so that* (22) *holds for all non–empty knot spans $]t^\ell_r, t^\ell_{r+1}[\in \Pi^\ell$ and for $\ell = 0, \ldots, N-1$. Then all univariate B–spline bases Γ^ℓ of degree d satisfy assumption* (A6).

Proof We consider the restriction of a spline function $f \in V^\ell$ to a non–empty knot span $\pi =]a, b[\in \Pi^\ell$ with $a = t^\ell_r$, $b = t^\ell_{r+1}$. It can be represented both in the B–spline basis and in the Bernstein basis with respect to this knot interval,

$$f(u) = \sum_{i=r-d+1}^{r+1} c_i \gamma_i(u) = \underbrace{\sum_{j=0}^{d} k_j b_j(u)}_{(BB)}, \quad u \in \pi. \tag{28}$$

In the first representation, each basis function is a B–spline γ_i with knots $t^\ell_{i-1}, \ldots, t^\ell_{i+d}$. For the second one, we use the Bernstein polynomials of degree d on π,

$$b_j(u) = \binom{d}{j} \left(\frac{u-a}{b-a}\right)^j \left(\frac{b-u}{b-a}\right)^{d-j}. \tag{29}$$

The coefficients c_i of the B–splines can be obtained by evaluating the blossom (or polar form) F of f via

$$c_i = F(t^\ell_i, t^\ell_{i+1}, \ldots, t^\ell_{i+d-1}), \qquad i = r-d+1, \ldots, r+1, \tag{30}$$

see, e.g., [21]. The blossom F is obtained from the Bernstein-Bézier representation (BB) in (28) by replacing the Bernstein polynomials b_j with their blossoms

$$B_j(u_1, \ldots, u_d) = \frac{1}{j!\,(d-j)!} \sum_{\sigma\in\Sigma} \prod_{i=1}^{j} \left(\frac{u_{\sigma(i)}-a}{b-a}\right) \prod_{i=j+1}^{d} \left(\frac{b-u_{\sigma(i)}}{b-a}\right), \tag{31}$$

where Σ denotes the set of all permutations of $\{1, \ldots, d\}$. Due to assumption (22), for all arguments $u_1, \ldots u_d$ satisfying $t^\ell_{r-d+1} \leq u_i \leq t^\ell_{r+d}$, the value of $B_j(u_1, \ldots, u_d)$ is bounded by $\binom{d}{j}T^d$. Consequently, the B–spline coefficients in (30) satisfy

$$|c_i| \leq (d+1)\binom{d}{j}T^d \max_{j=0,\ldots,d} |b_j|, \qquad i = r-d+1, \ldots, r+1. \tag{32}$$

Due to the stability of the Bernstein basis[4] on π there exists another constant K' which depends only on the degree d such that

$$|b_j| \leq K' \|f\|_{\infty,\pi}, \qquad j = 0, \ldots, d. \tag{33}$$

Combining the two inequalities (32)–(33) gives (A6) with $K = (d+1)\binom{d}{j}T^d K'$. □

For instance, if uniform knot vectors are used at all levels, then condition (22) is satisfied with $T = 2d - 1$.

In view of Theorem 19, when the hypotheses of Lemma 22 are satisfied, it follows that the truncated hierarchical B–splines obtained for any choice of subdomains $\Omega^0, \ldots, \Omega^{N-1}$ respecting (D) are strongly stable with respect to the L_∞ norm.

C A counterexample for strong stability of the hierarchical basis for multivariate tensor–product B–splines of any degree

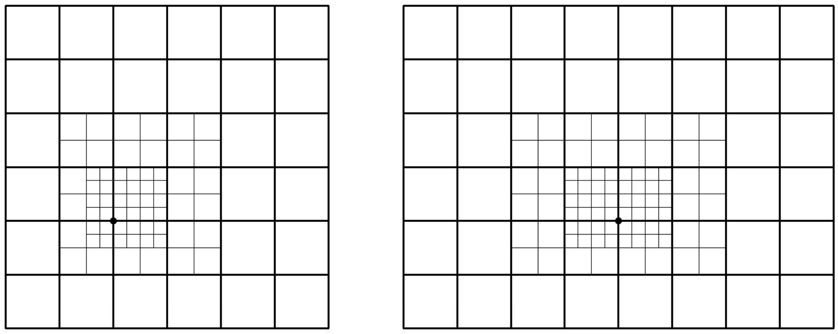
Let V^ℓ, $\ell = 0, \ldots, N-1$, be the n–variate tensor–product spline spaces of multi–degree $\mathbf{d} = (d_1, \ldots, d_n)$ with $d_i > 1$ defined on a grid spanned by uniformly spaced knot sequences in each direction where the knot interval is equal to $2^{-\ell}$. We choose the sequence of nested open subdomains as

$$\Omega^\ell = \left] -2^{1-\ell}\left\lceil\frac{d_1}{2}\right\rceil, \; 2^{1-\ell}\left(d_1 + 1 - \left\lceil\frac{d_1}{2}\right\rceil\right)\right[\times \ldots \times \left] -2^{1-\ell}\left\lceil\frac{d_n}{2}\right\rceil, \; 2^{1-\ell}\left(d_n + 1 - \left\lceil\frac{d_n}{2}\right\rceil\right)\right[.$$

Two sequences of this kind are shown in Figure 15. On each level ℓ, there is just one B–spline $\gamma^\ell \in V^\ell$ whose support is completely in the finer subdomain $\Omega^{\ell+1}$ (see again Figure 15). Hence, this B–spline will be eliminated from the hierarchical B–spline basis and replaced by finer ones. Let $0 < g_0 < 1$ be the value of this B–spline evaluated at the origin $\mathbf{0} = (0, 0 \ldots, 0)$, i.e.,

$$g_0 = \gamma^\ell(\mathbf{0}). \tag{34}$$

[4] The stability of the univariate Bernstein basis was analyzed in [8]. Specific bounds for K' in the case of univariate and multivariate triangular Bernstein bases were provided in [17].



(a) $(d_1, d_2) = (2, 2)$ (b) $(d_1, d_2) = (3, 2)$

Fig. 15 Counterexamples for strong stability of hierarchical B–splines.

Note that this value is independent of the choice of ℓ.

We now consider the spline function defined on Ω^ℓ represented in the hierarchical basis $\mathcal{H}$ with all coefficients equal to one,

$$f(\mathbf{u}) = \sum_{\gamma\in\mathcal{H}} c_\gamma\, \gamma(\mathbf{u}), \qquad c_\gamma = 1.$$

We may observe that the sequence of subdomains considered here ensures that at each level ℓ, all the B–splines of that level acting on the origin (except the one that is removed) are included in the hierarchical basis. By evaluating the spline f at the origin and taking (34) into account, we get

$$f(\mathbf{0}) = \sum_{\ell=0}^{N-1}(1-\gamma^\ell(\mathbf{0})) + \gamma^{N-1}(\mathbf{0}) = N(1-g_0) + g_0.$$

Hence,

$$\|f\|_{\infty,\Omega^0} \geq N(1-g_0) + g_0 = (N(1-g_0)+g_0)\max_{\gamma\in\mathcal{H}}|c_\gamma|.$$

We conclude that the value of k_1 in (26) has to grow at least linearly with the number of levels in the hierarchy. For instance one can consider $k_1 = N$.

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