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Mass Smoothers in Geometric Multigrid for Isogeometric Analysis

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Abstract. We investigate geometric multigrid methods for solving the large, sparse linear systems which arise in isogeometric discretizations of elliptic partial differential equations. In particular, we study a smoother which incorporates the inverse of the mass matrix as an iteration matrix, and which we call mass-Richardson smoother. We perform a rigorous analysis in a model setting and perform some numerical experiments to confirm the theoretical results.

1 Introduction

Isogeometric analysis (IGA), a numerical technique for the solution of partial differential equations first proposed in [11], has attracted considerable research attention in recent years. The use of spline spaces both for representation of the geometry and for approximation of the solution affords the method several very interesting features, such as the possibility to use exactly the geometry generated by CAD systems, refinement without further communication with the CAD system, the possibility of using high-continuity trial functions, the use of high-degree spaces with comparatively few degrees of freedom, and more. We refer to [11,1] as well as the monograph [5] and the references therein for an overview of the topic.

The efficient solution of the discretized systems arising in isogeometric analysis has been the topic of several publications in recent years, e.g., [4,12,6,7,3,8]. In the present paper, our interest lies in geometric multigrid methods for isogeometric analysis. It is known by now that the simple classical multigrid smoothers do not result in multigrid solvers with convergence rates which are robust in the spline degree of the IGA discretization. In this paper, motivated by promising results of a preliminary local Fourier analysis, we propose a smoother based on inverting the mass matrix. We perform a rigorous analysis in a model setting and find that, due to boundary effects, the smoother does not achieve total robustness in the spline degree with a single smoothing step, but requires a number of additional smoothing steps. We are able to quantify the needed number of smoothing steps for robust convergence. To our knowledge, this is the first rigorous analysis of a multigrid method for IGA which takes the spline degree into account explicitly.

The remainder of the paper is structured as follows. In Section 2, we outline a simple model problem and a geometric multigrid solver for IGA. In Section 3, we perform the analysis of the two-grid method with our proposed mass-Richardson smoother. In Section 4, we report the results of some numerical experiments which confirm the theory.

2 Geometric multigrid for isogeometric analysis

For a detailed description of the IGA methodology, see, e.g., [11]. For the sake of simplicity, we consider here only a simple model problem with a trivial geometry map. Previous numerical experiments indicate that nontrivial but well-behaved geometry maps do not significantly impact the convergence behaviour of multigrid methods for IGA.

Let $\mathcal{V}_h \subset H_0^1(\Omega)$ denote a tensor product B-spline space over $\Omega = (0,1)^d$. We use here spline spaces with open knot vectors which have the same spline degree p in each coordinate direction, and which have the same smoothness parameter k in each coordinate direction, where $k \in \{1, \dots, p\}$ describes splines which are globally C^{k-1}. We also assume that the spline spaces are quasi-uniform in the sense that the minimum knot span in any direction can be bounded from below by some uniform constant times the maximum knot span.

We consider an IGA discretization of the Poisson equation with pure Dirichlet boundary conditions: find $u_h \in \mathcal{V}_h$ such that

$$a(u_h, v_h) = \langle F, v_h \rangle \qquad \forall v_h \in \mathcal{V}_h$$

with the bilinear form and linear functions, respectively,

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \qquad \langle F, v \rangle = \int_\Omega f v \, dx - a(\tilde{g}, v).$$

Here $\tilde{g} \in H^1(\Omega)$ is a suitable extension of the given Dirichlet data.

In the following, we outline the construction of a simple geometric multigrid scheme for this problem. Let $\mathcal{V}_0$ denote a coarse tensor product spline space over $(0,1)^d$. Performing uniform and global h-refinement by knot insertion, we obtain a sequence of refined spline spaces $\mathcal{V}_1, \mathcal{V}_2, \dots$ Let $\mathcal{V}_H$ and $\mathcal{V}_h$ denote two successive spline spaces in this sequence, and let $P : \mathcal{V}_H \to \mathcal{V}_h$ denote the prolongation operator from the coarse to the fine space. One step of the two-grid iteration process is given by a pre-smoothing step, the coarse-grid correction, and a post-smoothing step; i.e., given $u_0 \in \mathcal{V}_h$, the next iterate u_1 is obtained by

$$\begin{aligned} u^{(1)} &:= u_0 + S^{-1}(f_h - A_h u_0), \\ u^{(2)} &:= u^{(1)} + P A_H^{-1} P^\top (f_h - A_h u^{(1)}), \\ u_1 := u^{(3)} &:= u^{(2)} + S^{-\top}(f_h - A_h u^{(2)}). \end{aligned}$$

Here, S is a suitable smoother for the fine-space stiffness matrix A_h.

As usual, a multigrid scheme is obtained by considering a hierarchy of nested spline spaces and replacing the exact inverse A_H^{-1} in the above procedure recursively with the same procedure applied on the next coarser space, until $\mathcal{V}_0$ is reached. We consider only the case of a single coarse-grid correction step, i.e., the V-cycle.

It is known that this multigrid algorithm with standard smoothers like the Richardson, Jacobi or Gauss-Seidel smoothers is robust in the mesh size h. However, the same is not true for the spline degree p: especially in higher space dimensions d, the iteration numbers for obtaining a desired accuracy increase strongly with p. This soon results in iteration numbers which are no longer practical.

It is therefore of interest to find smoothers which result in better or even completely robust iteration numbers with respect to p. An interesting choice is what we will call the mass-Richardson smoother,

$$S = \tau^{-1} M,$$

where M denotes the mass matrix and τ a real damping parameter. Local Fourier analysis suggests that this smoother should lead to convergence rates which are independent of p. However, as we will see, in actual boundary value problems a certain dependence on p remains due to boundary effects.

We note that applying this smoother requires inverting the isogeometric mass matrix in every iteration. Although the mass matrix is itself ill-conditioned for higher spline degrees p, an efficient approach for inverting it has been described in [9] by exploiting the tensor product structure of the spline spaces.

3 Analysis

We follow the ideas of the multigrid convergence theory as given by Hackbusch [10]. For simplicity, we restrict ourselves to the analysis of the two-grid method in either one or two space dimensions. From this, the convergence of the W-cycle follows by a perturbation argument, but the convergence of the V-cycle needs a different proof technique. Nevertheless, in practice we have observed that the V-cycle performs similarly to the two-grid method.

For any function u defined on the fine grid, we will write $\underline{u}$ for its coefficient vector in the fine-grid B-spline basis. In addition to the L_2-norm

$$\|u\|_0 := \|u\|_{L_2(\Omega)} = \|\underline{u}\|_M,$$

we will make use of the H^2-like norm

$$\begin{aligned} |\!|\!| u |\!|\!|_2 &:= \sup_{w \in V} \frac{a(u,w)}{\|w\|_0} = \sup_{\underline{w}} \frac{\underline{w}^\top A \underline{u}}{(\underline{w}^\top M \underline{w})^{1/2}} = \sup_{\underline{w}} \frac{\underline{w}^\top M^{-1/2} A \underline{u}}{(\underline{w}^\top \underline{w})^{1/2}} \\ &= \sup_{\|\underline{w}\|=1} \underline{w}^\top M^{-1/2} A \underline{u} = \|M^{-1/2} A \underline{u}\| = \|A\underline{u}\|_{M^{-1}} =: |\!|\!| \underline{u} |\!|\!|_2. \end{aligned}$$

By C, we will denote a generic constant which does not depend on the discretization parameters h, p and k.

For purposes of the analysis, we will always make the choice $\tau = 1/\lambda_{\max}(M^{-1}A)$ for the damping parameter.

We point out that we make use of the explicit spline approximation error estimates presented in [2]. So far, these results are limited to the case of relatively low-continuity splines, and this limitation therefore extends to our work.

3.1 Smoothing property

We will make use of the following polynomial inverse inequalities or Markov-type inequalities (see, e.g., Schwab [13]).

Theorem 1. *Let $d = 1$, $I = (a, b)$ and $h = b - a$, then for any polynomial f of degree at most p, we have*

$$\|f'\|_{L_2(I)} \leq 2\sqrt{3}\frac{p^2}{h}\|f\|_{L_2(I)}$$

Let $d = 2$. For an arbitrary quadrilateral $I = (a, b)\times(c, d)$ with size $h_1 = b-a$, $h_2 = d - c$, setting $h := \max\{h_1, h_2\}$ and assuming $h \leq C\min\{h_1, h_2\}$ with a uniformly bounded constant C, we have

$$\|\nabla f\|_{L_2(I)} \leq C\frac{p^2}{h}\|f\|_{L_2(I)}. \tag{1}$$

Theorem 2. *After ν steps of mass-Richardson smoothing, the resulting error $\underline{e}^{(\nu)} = (I - \tau M^{-1}A)^\nu \underline{e}$ satisfies*

$$\|\underline{e}^{(\nu)}\|_2 \leq \frac{Cp^4}{h^2\nu}\|\underline{e}^{(0)}\|_M.$$

Proof. By iterating the statement of Theorem 1 over every non-empty knot span of the spline space, we obtain for any $f \in C(\Omega) \cap L_2(\Omega)$ which is piecewise a polynomial of maximum degree at most p, and in particular for $f \in \mathcal{V}_h$,

$$\|\nabla f\|_{L_2(\Omega)} \leq C\frac{p^2}{h}\|f\|_{L_2(\Omega)}.$$

It follows that $\langle A\underline{v}, \underline{v}\rangle \leq C\frac{p^4}{h^2}\langle M\underline{v}, \underline{v}\rangle$ and therefore $\lambda_{\max}(M^{-1}A) \leq C\frac{p^4}{h^2}$.

We have

$$\begin{aligned}
\|A\underline{e}^{(\nu)}\|_{M^{-1}} &= \|A(I - \tau M^{-1}A)^\nu \underline{e}\|_{M^{-1}} \\
&= \|M^{-1}A(I - \tau M^{-1}A)^\nu \underline{e}\|_M \\
&= \tfrac{1}{\tau}\|\tau M^{-1}A(I - \tau M^{-1}A)^\nu \underline{e}\|_M \\
&= \tfrac{1}{\tau}\|\tau M^{-1/2}AM^{-1/2}(I - \tau M^{-1/2}AM^{-1/2})^\nu M^{1/2}\underline{e}\| \\
&\leq \tfrac{1}{\tau}\|X(I - X)^\nu\|\|\underline{e}\|_M \\
&= \tfrac{1}{\tau}\max\{\lambda(1 - \lambda)^\nu : \lambda \in \sigma(X)\}\|\underline{e}\|_M
\end{aligned}$$

with the symmetric matrix $X = \tau M^{-1/2} A M^{-1/2}$ which has the same spectrum as $\tau M^{-1} A$. Thus, $\sigma(X) \subset [0,1]$, and with the estimate $\lambda(1-\lambda)^\nu \le 1/(e\nu)$ for $\lambda \in [0,1]$, we obtain

$$\|A\underline{e}^{(\nu)}\|_{M^{-1}} \le \frac{C\lambda_{\max}(M^{-1}A)}{\nu}\|\underline{e}\|_M \le \frac{Cp^4}{h^2\nu}\|\underline{e}\|_M. \qquad \square$$

3.2 Approximation property

We summarize some recent results from [2] on the approximation properties of spline spaces with explicit dependence on h, p and k.

Theorem 3. *Assume that, if $d = 1$, then $k \le \sigma \le p+1$, or if $d = 2$, then $2k \le \sigma \le p+1$. There exists a spline interpolation operator $\Pi : H^\sigma(\Omega) \to \mathcal{V}_h$ such that for all $v \in H^\sigma(\Omega)$ and $j = 0, \dots, \sigma$, we have*

$$|v - \Pi v|_{H^\ell(\Omega)} \le C(p-k+1)^{-(\sigma-\ell)} h^{\sigma-\ell} |v|_{H^\sigma(\Omega)}.$$

Theorem 4. *Under the assumptions of Theorem 3 and full H^2-regularity of the boundary value problem, the errors e before and e^{CGC} after the coarse-grid correction step satisfy*

$$\|\underline{e}^{CGC}\|_M \le Ch^2(p-k+1)^{-2}\|\!|\underline{e}|\!\|_2.$$

Proof. With $u_0 = u - e$ being the iterate before the coarse-grid correction step, the correction $t_C \in V_C$ such that $e^{\mathrm{CGC}} = e - t_C$ is given by

$$a(t_C, w_C) = \langle F, w_c\rangle - a(u_0, w_C) = a(e, w_C) \qquad \forall w_C \in V_C.$$

For an arbitrary bounded linear functional $F^* : L_2(\Omega) \to \mathbb{R}$, we introduce the solutions of the dual problems $\tilde{\xi} \in H_0^1(\Omega)$, $\xi \in V$, and $\xi_C \in V_C$, respectively, by

$$\begin{aligned} a(\tilde{x}, \tilde{\xi}) &= \langle F^*, \tilde{x}\rangle \qquad \forall \tilde{x} \in H_0^1(\Omega), \\ a(x, \xi) &= \langle F^*, x\rangle \qquad \forall x \in V, \\ a(x_C, \xi_C) &= \langle F^*, x_C\rangle \qquad \forall x_C \in V_C. \end{aligned}$$

We have the identity

$$\langle F^*, e - t_C\rangle = a(e - t_C, \xi) = a(e, \xi) - \langle F^*, t_C\rangle = a(e, \xi - \xi_C)$$

and thus

$$|\langle F^*, e - t_C\rangle| \le \frac{|a(e, \xi - \xi_C)|}{\|\xi - \xi_C\|_0}\|\xi - \xi_C\|_0 \le \sup_{w \in V} \frac{|a(e,w)|}{\|w\|_0}\|\xi - \xi_C\|_0.$$

Using Theorem 3 and a standard Nitsche duality argument under the assumption of full H^2-regularity, we find that

$$\|\xi - \xi_C\|_0 \le \|\xi - \tilde{\xi}\|_0 + \|\tilde{\xi} - \xi_C\|_0 \le Ch^2(p-k+1)^{-2}\|F^*\|_*.$$

From the above two inequalities, it follows that

$$\|e^{\mathrm{CGC}}\|_0 = \|e - t_C\|_0 = \sup_{F^*} \frac{|\langle F^*, e - t_C\rangle|}{\|F^*\|_*} \le Ch^2(p-k+1)^{-2} \sup_{w \in V} \frac{a(e,w)}{\|w\|_0}. \quad \square$$

3.3 Two-grid convergence result

Theorem 5. *Under the assumptions of Theorem 4, the errors $e^{(0)}$ and $e^{(1)}$ before and after one two-grid cycle with ν mass-Richardson presmoothing steps and no postsmoothing satisfy*

$$\|e^{(1)}\|_0 \leq \frac{Cp^4}{\nu(p-k+1)^2}\|e^{(0)}\|_0.$$

In particular, there exists a positive constant C_ν independent of h, p and k such that the choice

$$\nu \geq C_\nu p^2$$

for the number of smoothing steps guarantees a convergence rate $\sigma \in (0,1)$ such that $\|e^{(1)}\|_0 \leq \sigma\|e^{(0)}\|_0$ with σ independent of h, k and p.

Proof. The first estimate follows directly by combining the statements of Theorem 2 and Theorem 4. By the assumptions of Theorem 4, we can conclude $p-k+1 \geq p/2$ and hence $p^4/(p-k+1)^2 \leq Cp^2$, which proves the second statement. □

4 Numerical experiments

We solve the described model Poisson problem with an exact solution $u(x) = \prod_{i=1}^{d} \sin(\pi(x_i + 0.5))$ using two-grid iteration. We start with a random starting vector and iterate until the initial residual is reduced by a factor of 10^{-8} in the ℓ_2-norm. The obtained iteration numbers are more or less independent of the mesh size, and we do therefore not report numbers for different h. The problem sizes were relatively small with at most a few thousand degrees of freedom.

p	k	τ	ν	iter.	ν	iter.
1	1	0.13	1	34	1	34
2	2	0.16	4	9	2	17
3	2	0.16	9	5	3	13
4	2	0.15	16	4	4	11
5	2	0.14	25	4	5	10
6	2	0.13	36	3	6	13

Table 1. Iteration numbers in 1D. From left to right: spline degree p, smoothness parameter k, damping parameter τ; iteration numbers for $\nu = p^2$, $\nu = p$ smoothing steps.

Both in 1D (Table 1) and in 2D (Table 2), the iteration numbers for p^2 smoothing steps remain uniformly bounded as we increase p. This confirms the theory with a choice of the constant $C_\nu = 1$ from Theorem 5. For comparison,

p	k	τ	ν	iter.	ν	iter.
1	1	0.08	1	65	1	65
2	2	0.10	4	15	2	30
3	2	0.09	9	8	3	23
4	2	0.09	16	5	4	44
5	2	0.08	25	4	5	16

Table 2. Iteration numbers in 2D. From left to right: spline degree p, smoothness parameter k, damping parameter τ; iteration numbers for $\nu = p^2$, $\nu = p$ smoothing steps.

we have also included the choice $\nu = p$, which is not supported by theory but serves as an interesting comparison.

We do not include CPU times here as we have not implemented the optimal mass matrix inversion algorithm from [9] at present, and therefore the results would be skewed towards lower smoothing numbers. We also note that the iteration numbers can be reduced, often significantly so, by replacing the two-grid iteration with CG iteration preconditioned by one two-grid cycle.

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