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Analysis of Discontinuous Galerkin IGA Approximations to Elliptic Boundary Value Problems

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Abstract. *In this work, we study the approximation properties of a new method, that applies the Isogeometric Analysis (IGA) discretization concept and the Discontinuous Galerkin technique on the interfaces, for solving linear diffusion with discontinuous diffusion coefficients. The computational domain is divided into non-overlapping sub-domains, called patches in IGA, where* $\mathbb{B}-$*Spline finite dimensional approximations spaces are constructed. The solution of the problem is approximated in every sub-domain without imposing any matching grid conditions and without any continuity requirements for the discrete solution on the interfaces. Numerical fluxes with interior penalty jump terms are applied in order to treat the discontinuities of the discrete solution on the interfaces. We present an a priori error analysis for problems set in two- and three- dimensional domains, with solutions belonging to* $W^{l,p}, l \geq 2,\ p \in (\frac{2d}{d+2(l-1)}, 2]$. *In any case, we show optimal convergence rates of the discretization with respect to* $\|.\|_{DG}$*-norm.*

Key words: linear elliptic problems, discontinuous coefficients, Discontinuous Galerkin discretization, Isogeometric Analysis, non-matching grids, low regularity solutions, a priori error estimates.

1 Introduction

The finite element methods (FEM) and, in particular, discontinuous Galerkin (DG) finite element methods are very often used for solving elliptic boundary value problems which arise from engineering applications, see, e.g., [1],[2],[3]. Although the isoparametric FEM and even FEM with curved finite elements have been proposed and analyzed long time ago, cf. [4], [5], [6], [1], the quality of the numerical results for realistic problems in complicated geometries depends on the quality of the discretized geometry (triangulation of the domain), which is usually performed by a mesh generator. In many situations (e.g. fluid dynamics problems), extremely fine meshes are required around objects, singular corner points e.t.c. in order to achieve numerical solutions with desired resolution. This fact leads to an increased number of degrees of freedom, and thus to an increased overall computational cost for solving the discrete problem, see, e.g., [7].

Recently, the Isogeometric Analysis (IGA) concept has been applied for approximating solutions of elliptic problems [8], [9]. IGA generalizes and improves the classical FE (even isoparametric FE) methodology in the following direction: complex computational domains can be exactly represented as images of parametric functions which are constructed by using superior classes of finite dimensional spaces e.g. $\mathbb{B}$-Splines, Non-Uniform Rational $\mathbb{B}$-Splines (NURBS), see [10], [11]. The same class of functions is used to approximate the exact solution without increasing the computational cost for the computation of the resulting stiffness matrices [12], systematic $h-k$ refinement procedures can easily be developed [13], and, last but not least, the method can be materialized in parallel environment incorporating fast domain decomposition solvers [14], [15], [16].

During the last two decades, there has been an increasing interest in discontinuous Galerkin finite element methods for the numerical solution of several types of partial differential equations, see, e.g., [17] and [2]. This is due to the advantages of the local approximation spaces without continuity requirements that DG methods offer [18], [19], [20], [21].

In this paper, we develop a method by trying to combine the best features of the two aforementioned methods. Specifically, we study and analyze the IGA approximation properties to elliptic boundary value problems with discontinuous coefficients. The problem is set in a complex, bounded Lipschitz domain $\Omega \subset \mathbb{R}^d, d = 2, 3$, which is subdivided in a union of non-overlapping sub-domains, say $\mathcal{S}(\Omega) := \{\Omega_i\}_{i=1}^N$. For simplicity, we assume that the discontinuity of the diffusion coefficients is only observed across sub-domain boundaries (interfaces). The weak solution of the problem is approximated in every sub-domain applying IGA methodology, [9], without matching grid conditions along the $\partial\Omega_i$, as well without imposing continuity requirements for the approximation spaces on $\partial\Omega_i$. By construction, DG methods use discontinuous approximation spaces utilizing numerical fluxes on the interfaces, [22], and have been efficiently used for solving problems on non-matching grids in the past, [21], [23], [24]. Here, the numerical scheme is formulated by applying numerical fluxes with interior penalty coefficients on the interfaces of the sub-domains (patches), and using IGA in every patch independently. The resulting discretization technique is called Discontinuous Galerkin Isogeometric Analysis (DGIGA). A crucial point in the presented work, is the expression of the numerical flux interface terms as a sum over the micro-elements edges taking note of the non-matching sub-domain grids. This gives the opportunity to proceed in the error analysis by applying the trace inequalities locally as in DG finite element methods. There are many papers, which present DG finite element approximations for elliptic problems, see, e.g., [18], [25], the monographs [20],[19], and, in particular, for the discontinuous coefficient case, [21], [26]. However, there are only a few publications on the DGIGA and their analysis. In [27], the author presented discretization error estimates for the DGIGA of plane (2d) diffusion problems on meshes matching across the patch boundaries and under the assumption of sufficiently smooth solutions. This analysis obviously carries over to plane linear elasticity problems which have recently been studied numerically in [16]. In [28], the DG technology has been used to handle no-slip boundary conditions and multi-patch geometries for IGA of Darcy-Stokes-Brinkman equations. DGIGA discretizations of heterogenous diffusion problems on open and closed surfaces, which are given by a multipatch NURBS representation, are constructed and rigorously analysed in [29].

In the first part, we give a priori error estimates in the $\|.\|_{DG}$ norm under the usual regularity assumption on the exact solution, i.e. $u \in W^{1,2}(\Omega) \cap W^{l\geq 2,2}(\mathcal{S}(\Omega))$. Next, we consider the model problem with low regularity solution $u \in W^{1,2}(\Omega) \cap W^{l\geq 2,p\in(\frac{2d}{d+2(l-1)},2)}(\mathcal{S}(\Omega))$ and derive error estimates in the $\|.\|_{DG}$. These estimates are optimal with respect to the space size discretization. We note that the error analysis in the case of low regularity solutions includes many ingredients of the DG FE error analysis of [30] and [26] on low regularity boundary value problems. To the best of our knowledge, optimal error analysis for IGA discretizations combined with DG techniques for solving elliptic problems with discontinuous coefficients in general domains $\Omega \subset \mathbb{R}^d, d = 2, 3$ have not been yet presented in the literature.

The paper is organized as follows. In Section 2 the PDE problem is described. In Section 3, we introduce some notations. The local $\mathbb{B}_h(\mathcal{S}(\Omega))$ approximation space and the numerical scheme are also presented. Several auxiliary results and the analysis of the method for the case of usual regularity solutions are provided in Section 4. The analysis of the method for low regularity solutions is given in Section 5. Section 6 includes several numerical examples that verify the theoretical convergence rates. The paper closes with the conclusions.

2 The model problem

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d,\ d=2,3$, with the boundary $\partial\Omega$. For simplicity, we restrict our study to the model problem

$$-\mathrm{div}(\alpha\nabla u) = f \text{ in } \Omega, \text{ and } u = u_D \text{ on } \partial\Omega, \tag{2.1}$$

where f and u_D are given smooth data. In (2.1), α is the diffusion coefficient and assume be bounded by above and below by strictly positive constants.

The weak formulation is to find a function $u \in W^{1,2}(\Omega)$ such that $u := u_D$ on $\partial\Omega$ and satisfies

$$a(u,\phi) = l(\phi),\ \forall\phi \in W_0^{1,2}(\Omega), \tag{2.2a}$$

where

$$a(u,\phi) = \int_\Omega \alpha\nabla u\nabla\phi\,dx, \quad \text{and} \quad l(\phi) = \int_\Omega f\phi\,dx. \tag{2.2b}$$

Results concerning the existence and uniqueness of the solution u of problem (2.2) can be derived by a simple application of Lax-Milgram Lemma, [31]. To avoid unnecessary long formulas below, we only considered in (2.1) non-homogeneous Dirichlet boundary conditions on $\partial\Omega$. However, the analysis can be easily generalized to Neumann and Robin type boundary conditions on a part of $\partial\Omega$, since they are naturally introduced in the DG formulation.

3 Preliminaries - DG notation

Throughout this work, we denote by $L^p(\Omega), p > 1$ *the Lebesgue spaces* for which $\int_\Omega |u(x)|^p\,dx < \infty$, endowed with the norm $\|u\|_{L^p(\Omega)} = \big(\int_\Omega |u(x)|^p\,dx\big)^{\frac{1}{p}}$. By $\mathcal{D}(\Omega)$, we define the the space of C^∞ functions with compact support in Ω, and by $C^k(\Omega)$ the set of functions with $k-th$ order continues derivatives. In dealing with differential operators in Sobolev spaces, we use the following common conventions. For any (multi-index) $\alpha = (\alpha_1,...,\alpha_d),\ \alpha_j \geq 0, j=1,...,d$, with degree $|\alpha| = \sum_{j=1}^d \alpha_j$, we define the *differential operator*

$$D^a = D_1^{\alpha_1}\cdots D_d^{\alpha_d}, \text{with } D_j = \frac{\partial}{\partial x_j}, D^{(0,...,0)}u = u. \tag{3.1}$$

We also denote by $W^{l,p}(\Omega)$, l positive integer and $1 \leq p \leq \infty$, the Sobolev space functions endowed with the norm

$$\|u\|_{W^{l,p}(\Omega)} = \big(\sum_{0\leq|\alpha|\leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \big)^{\frac{1}{p}}, \tag{3.2a}$$

$$\|u\|_{W^{l,\infty}(\Omega)} = max_{0\leq|\alpha|\leq m} \|D^\alpha u\|_\infty. \tag{3.2b}$$

For more details for the above definitions, we refer [32].

In order to apply the IGA methodology for the problem (2.1), the domain Ω is subdivided into a union polygonal sub-domains, $\mathcal{S}(\Omega) := \{\Omega_i\}_{i=1}^N$, such that

$$\bar{\Omega} = \bigcup_{i=1}^N \bar{\Omega}_i, \quad \text{with} \quad \Omega_i \cap \Omega_j = \emptyset, \text{ if } j \neq i. \tag{3.3}$$

The subdivision of Ω assumed to be compatible with the discontinuities of α, [21],[26]. In other words, the diffusion coefficient assumed to be constant in the interior of Ω_i and its discontinuities can appear only on the interfaces $F_{ij} = \partial\Omega_i \bigcap \partial\Omega_j$.

As it is common in the IGA analysis, we assume a parametric domain $\widehat{D}$ of unit length, (e.g. $\widehat{D} = [0,1]^d$). For any Ω_i, we associate $n = 1, ..., d$ knot vectors $\Xi_n^{(i)}$ on $\widehat{D}$, which create a mesh $T^{(i)}_{h_i,\widehat{D}} = \{\hat{E}_m\}_{m=1}^{M_i}$, where $\hat{E}_m$ are the micro-elements. We shall refer $T^{(i)}_{h_i,\widehat{D}}$ as the *parametric mesh of* Ω_i. For all $\hat{E}_m \in T^{(i)}_{h_i,\widehat{D}}$ we denote by $h_{\hat{E}_m}$ *the diameter of* $\hat{E}_m$ and by $h_i = \max\{h_{\hat{E}_m}\}$ *the parametric meshsize* of $T^{(i)}_{h_i,\widehat{D}}$. Further, we assume the following properties on every $T^{(i)}_{h_i,\widehat{D}}$,

- if $\hat{E}_m \in T^{(i)}_{h_i,\widehat{D}}$ then $h_i \sim h_{\hat{E}_m}$, and also for the boundary edges $e_{\hat{E}_m} \subset \partial\hat{E}_m$ holds that $h_{\hat{E}_m} \sim e_{\hat{E}_m}$,
- quasi-uniformity: for two adjacent micro-elements $\hat{E}_m, \hat{E}_n$ it holds $h_{\hat{E}_m} \sim h_{\hat{E}_n}$,

On every $T^{(i)}_{h_i,\widehat{D}}$, we construct the finite dimensional space $\hat{\mathbb{B}}^{(i)}_{h_i}$ spanned by $\mathbb{B}$-Spline basis functions of degree k, [10],

$$\hat{\mathbb{B}}^{(i)}_{h_i} = span\{\hat{B}^{(i)}_j(\hat{x})\}_{j=0}^{dim(\hat{\mathbb{B}}^{(i)}_{h_i})}, \tag{3.4a}$$

where every j-base function $\hat{B}_j^{(i)}(\hat{x})$ in (3.4a) is derived by means of tensor products of one-dimensional $\mathbb{B}$-Spline basis functions, e.g.

$$\hat{B}_j^{(i)}(\hat{x}) = \hat{B}_{j_1}^{(i)}(\hat{x}_1) \cdots \hat{B}_{j_d}^{(i)}(\hat{x}_d). \tag{3.4b}$$

For simplicity in the following error analysis, we consider the case where the basis functions of every $\hat{\mathbb{B}}_{h_i}^{(i)}, i = 1, ..., N$ have the same degree, k.

Every sub-domain $\Omega_i \in \mathcal{S}(\Omega), i = 1, ..., N$, is exactly represented through a parametrization (one-to-one mapping), [12], having the form

$$\mathbf{\Phi}_i : \widehat{D} \to \Omega_i, \quad \mathbf{\Phi}_i(\hat{x}) = \sum_j C_j^{(i)} \hat{B}_j^{(i)}(\hat{x}) := x \in \Omega_i, \tag{3.5a}$$

$$\text{with} \quad \hat{x} = \mathbf{\Psi}_i(x) := \mathbf{\Phi}_i^{-1}(x), \tag{3.5b}$$

where $C_j^{(i)}$ are the control points.Based on $\mathbf{\Phi}_i$ of (3.5), we construct a mesh $T_{h_i,\Omega_i}^{(i)} = \{E_m\}_{m=1}^{M_i}$ for every Ω_i, whose vertices are the images of the vertices of the corresponding parametric mesh $T_{h_i,\widehat{D}}^{(i)}$ through $\mathbf{\Phi}_i$. If $h_{\Omega_i} = \max\{h_{E_m}\}$, $E_m \in T_{h_i,\Omega_i}^{(i)}$ is the sub-domain Ω_i mesh size, then based on definition (3.5) of $\mathbf{\Phi}_i$, there is a constant $C := C(\|\mathbf{\Phi}_i(\hat{x})\|_\infty)$ such that $h_i \sim C h_{\Omega_i}$. In what follows, we denote the sub-domain mesh size by h_i without the constant $C := C(\|\mathbf{\Phi}_i(\hat{x})\|_\infty)$ explicitly appearing.

The mesh of Ω can be considered to be $T_h(\Omega) = \bigcup_{i=1}^N T_{h_i,\Omega_i}^{(i)}$, where we note that there are no matching mesh requirements on the interior interfaces $F_{ij} = \partial\Omega_i \bigcap \partial\Omega_j, i \neq j$. For the sake of brevity in our notations, the interior faces of the boundary of the sub-domains are denoted by $\mathcal{F}_I$ and the collection of the faces that belong to $\partial\Omega$ by $\mathcal{F}_B$, e.g. $F \in \mathcal{F}_B$ if there is a Ω_i such that $F = \partial\Omega_i \bigcap \partial\Omega$. We denote the set of all sub-domain faces by $\mathcal{F}$.

Further, we define on Ω the finite dimensional $\mathbb{B}-$Spline space $\mathbb{B}_h(\mathcal{S}(\Omega)) = \mathbb{B}_{h_1}^{(1)} \times ... \times \mathbb{B}_{h_N}^{(N)}$, where every $\mathbb{B}_{h_i}^{(i)}$ is defined on $T_{h_i,\Omega_i}^{(i)}$ as follows

$$\mathbb{B}_{h_i}^{(i)} := \{B_{h_i}^{(i)}|_{\Omega_i} : B_h^{(i)}(\hat{x}) = \hat{B}_h^{(i)} \circ \mathbf{\Psi}_i(x), \ \forall \hat{B}_h^{(i)} \in \hat{\mathbb{B}}_{h_i}^{(i)}\}. \tag{3.6}$$

For every $\hat{E} \in T_{h_i,\widehat{D}}^{(i)}$, we denote by $\tilde{D}_{\hat{E}}^{(i)}$ its support extension, the union of all the basis function $\hat{B}_j^{(i)} \in \hat{\mathbb{B}}_{h_i}^{(i)}$ supports whose support intersects $\hat{E}$. Analogously, we define the union support in physical sub-domain Ω_i by $D_E^{(i)}$ to be the image of $\tilde{D}_{\hat{E}}^{(i)}$ through the paramterization map $\mathbf{\Phi}_i$, e.g. $D_E^{(i)} := \mathbf{\Phi}(\tilde{D}_{\hat{E}}^{(i)})$. Since $\mathbf{\Phi}_i(\hat{x}) \in \hat{\mathbb{B}}_h^{(i)}$, the components

$\Phi_{1,i},...,\Phi_{d,i} \in \hat{\mathbb{B}}_h^{(i)}$ are smooth functions and hence there exist constants c_m, c_M such that

$$c_m \leq |det(\mathbf{\Phi}_i^{'}(\hat{x}))| \leq c_M,\ i=1,...,d,\ \text{for all } \hat{x} \in \hat{D} \tag{3.7}$$

where $\mathbf{\Phi}_i^{'}(\hat{x})$ denotes the Jacobian matrix $\frac{\partial(x_1,...,x_d)}{\partial(\hat{x}_1,...,\hat{x}_d)}$.

Now, for any $\hat{u} \in W^{m,p}(\hat{D}), m \geq 0, p > 1$, we define the function

$$\mathcal{U}(x) = \hat{u}(\mathbf{\Psi}_i(x)),\ x \in \Omega_i, \tag{3.8}$$

where the mapping $\mathbf{\Psi}$ is defined in (3.5b). For the analysis presented below, it is necessary to show the following relation

$$C_m \|\hat{u}\|_{W^{m,p}(\hat{D})} \leq \|\mathcal{U}\|_{W^{m,p}(\Omega_i)} \leq C_M \|\hat{u}\|_{W^{m,p}(\hat{D})}, \tag{3.9}$$

where the constants C_m, C_M depending on

$$C_m := C_m(\max_{m_0 \leq m}(\|D^{m_0}\mathbf{\Phi}_i\|_\infty), \|det(\mathbf{\Psi}_i^{'}(\hat{x}))\|_\infty)$$

and

$$C_M := C_M(\max_{m_0 \leq m}(\|D^{m_0}\mathbf{\Psi}_i\|_\infty), \|det(\mathbf{\Phi}_i^{'}(\hat{x}))\|_\infty)$$

correspondingly.

Indeed, for any $\hat{u} \in W^{m,p}(\hat{D})$ we can find a sequence $\{\hat{u}_j\} \subset C^\infty(\bar{\hat{D}})$ converging to $\hat{u}$ in $\|.\|_{W^{m,p}(\hat{D})}$, we then have by applying the chain rule in (3.8) that

$$D_x(\mathbf{\Psi}_i(x))^{-1} D\mathcal{U}_j(x) = D\hat{u}_j(\mathbf{\Psi}_i(x)). \tag{3.10}$$

Then for any multi-index m we can get the following formula

$$D^m \mathcal{U}_j(x) = \sum_{m_0 \leq m} P_{m,m_0}(x) D^{m_0} \mathcal{U}_j(x), \tag{3.11}$$

where $P_{m,m_0}(x) \in \mathbb{B}_h^{(i)}$ is a polynomial of degree less than k and includes the various derivatives of $\mathbf{\Psi}_i(x)$. Multiplying (3.11) by $\varphi(x) \in \mathcal{D}(\Omega_i)$, and integrating by parts we obtain

$$(-1)^{|m|} \int_{\Omega_i} \mathcal{U}_j(x) D^m \varphi(x)\, dx = \sum_{m_0 \leq m} \int_{\Omega_i} P_{m,m_0}(x) D^{m_0} \mathcal{U}_j(x) \varphi(x)\, dx. \tag{3.12}$$

If we transfer the integral in (3.12) to integrals over $\hat{D}$ using the change of variable $x = \mathbf{\Phi}_i(\hat{x})$ we get

$$(-1)^{|m|} \int_{\hat{D}} \hat{u}_j(\hat{x}) D^m \varphi(\mathbf{\Phi}_i(\hat{x})) |det(\mathbf{\Phi}_i^{'}(\hat{x}))| \, d\hat{x} = \\ \sum_{m_0 \leq m} \int_{\hat{D}} P_{m,m_0}(\mathbf{\Phi}_i(\hat{x})) D^{m_0} \hat{u}_j(\hat{x}) \varphi(\mathbf{\Phi}_i(\hat{x})) |det(\mathbf{\Phi}_i^{'}(\hat{x}))| \, d\hat{x}. \quad (3.13)$$

But it holds that $D^{m_0}\hat{u}_j \to D^{m_0}\hat{u}$ in $\|.\|_{L^p(\hat{D})}$, thus taking the limit $j \to \infty$ in (3.13) and transferring the integrals back to Ω_i, we can derive (3.12) with respect to $\mathcal{U}$. We conclude that (3.11) holds in the distributional sense, and therefore

$$\int_{\Omega_i} |D^m \mathcal{U}(x)|^p \, dx \leq C_p \int_{\Omega_i} \sum_{m_0 \leq m} \left| P_{m,m_0}(x) D^{m_0} \mathcal{U}(x) \right|^p dx \leq \\ C_p \max_{m_0 \leq m} \left(\max_{x \in \Omega_i} (P_{m,m_0}(x)) \right) \sum_{m_0 \leq m} \int_{\Omega_i} \left| D^{m_0} \mathcal{U}(x) \right|^p dx \leq \\ C_p \max_{m_0 \leq m} \left(\max_{x \in \Omega_i} (P_{m,m_0}(x)) \right) \max_{\hat{x} \in \hat{D}} (|det(\mathbf{\Phi}_i^{'}(\hat{x}))| \sum_{m_0 \leq m} \int_{\hat{D}} \left| D^{m_0} \hat{u}(\hat{x}) \right|^p d\hat{x} \leq \\ C(\max_{m_0 \leq m} (\|D^{m_0} \mathbf{\Psi}_i(x)\|_\infty, \|det(\mathbf{\Phi}_i^{'}(\hat{x}))\|_\infty) \sum_{m_0 \leq m} \left| D^{m_0} \hat{u}(\hat{x}) \right|^p_{W^{m_0,p}(\hat{D})}. \quad (3.14)$$

The "right inequality" of (3.9) follows immediately by (3.14). The "left inequality" of (3.9) can be derived by using the reverse change of variable $\hat{x} = \mathbf{\Psi}_i(x)$ and following the same arguments as above.

3.1 The numerical scheme

We use the $\mathbb{B}-$Spline spaces $\mathbb{B}_h^{(i)}$ defined in (3.6) for approximating the solution of (2.2) in every sub-domain Ω_i. Continuity requirements for $\mathbb{B}_h(\mathcal{S}(\Omega))$ are not imposed on the interfaces F_{ij} of the sub-domains, clearly $\mathbb{B}_h(\mathcal{S}(\Omega)) \subset L^2(\Omega)$ but $\mathbb{B}_h(\mathcal{S}(\Omega)) \nsubseteq W^{1,2}(\Omega)$. Thus, the problem (2.2) is discretized by discontinuous Galerkin techniques on F_{ij}, [21]. Using the notation $\phi_h^{(i)} := \phi_h|_{\Omega_i}$, we define the average and the jump of ϕ_h on $F_{ij} \in \mathcal{F}_I$ respectively by

$$\{\phi_h\} := \frac{1}{2}(\phi_h^{(i)} + \phi_h^{(j)}), \; [\![\phi_h]\!] := \phi_h^{(i)} - \phi_h^{(j)}, \quad (3.15a)$$

and for $F_i \in \mathcal{F}_B$

$$\{\phi_h\} := \phi_h,\ [\![\phi_h]\!] := \phi_h^{(i)}. \tag{3.15b}$$

The DGIGA method has as follows: find $u_h \in \mathbb{B}_h(\mathcal{S}(\Omega))$ such that

$$a_h(u_h, \phi_h) = l(\phi_h) + p_D(u_D, \phi_h),\ \forall \phi_h \in \mathbb{B}_h(\mathcal{S}(\Omega)), \tag{3.16a}$$

where

$$a_h(u_h, \phi_h) = \sum_{i=1}^{N} a_i(u_h, \phi_h) - \sum_{F_{ij}\in\mathcal{F}} \frac{1}{2} s_i(u_h, \phi_h) + p_i(u_h, \phi_h), \tag{3.16b}$$

where the linear forms in (3.16b) are as follows, [21],

$$a_i(u_h, \phi_h) = \int_{\Omega_i} \alpha \nabla u_h \nabla \phi_h \, dx, \tag{3.16c}$$

$$s_i(u_h, \phi_h) = \int_{F_{ij}\in\mathcal{F}} \{\alpha \nabla u_h\} \cdot \mathbf{n}_{F_{ij}} [\![\phi_h]\!] \, ds, \tag{3.16d}$$

$$p_i(u_h, \phi_h) = p_{i_I}(u_h, \phi_h) + p_{i_B}(u_h, \phi_h) = \tag{3.16e}$$

$$\int_{F_{ij}\in\mathcal{F}_I} \Big(\frac{\mu\alpha^{(j)}}{h_j} + \frac{\mu\alpha^{(i)}}{h_i}\Big) [\![u_h]\!][\![\phi_h]\!] \, ds + \int_{F_i\in\mathcal{F}_B} \frac{\mu\alpha^{(i)}}{h_i} [\![u_h]\!][\![\phi_h]\!] \, ds,$$

$$p_D(u_D, \phi_h) = \int_{F_i\in\mathcal{F}_B} \frac{\mu\alpha^{(i)}}{h_i} u_D \phi_h \, ds, \tag{3.16f}$$

where the unit normal vector $\mathbf{n}_{F_{ij}}$ is oriented from Ω_i towards the interior of Ω_j and the parameter $\mu > 0$ will be specified later in the error analysis.

For notation convenience in what follows, we will use the same expression

$$\int_{F_{ij}} \Big(\frac{\mu\alpha^{(j)}}{h_j} + \frac{\mu\alpha^{(i)}}{h_i}\Big) [\![u_h]\!][\![\phi_h]\!] \, ds,$$

for both cases, $F_{ij} \in \mathcal{F}_I$ and $F_i \in \mathcal{F}_B$. In the later case we will assume that $\alpha^{(j)} = 0$.

We mention that in [21], apart from the form (3.16) another discrete formulation has been considered by introducing harmonic averages of the diffusion coefficients on the interface fluxes. Also, harmonic averages of the two different grid sizes have been used to penalize the jumps. Here, we prefer the forms (3.16d), (3.16e) for their simplicity. The possibility of using other averages for constructing the diffusion terms in front of the consistency and penalty terms has been analysed in several works, see e.g. [26], [33].

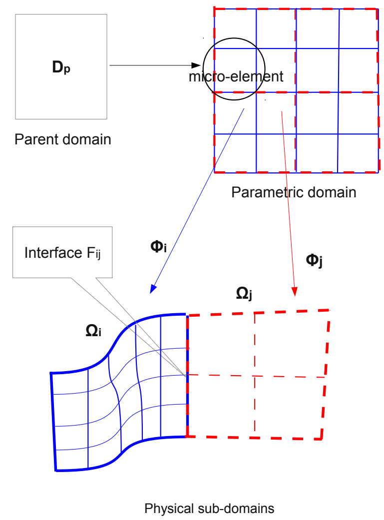

Fig. 1. The parent element, the parametric domain and two adjacent sub-domains.

4 Auxiliary results

In order to proceed to error analysis, several auxiliary results must be shown for $u \in W^{l,p}(\mathcal{S}(\Omega))$ and $\phi_h \in \mathbb{B}_h(\mathcal{S}(\Omega))$. The general frame of the proofs consists of three steps: (i) the required relations are expressed-proved on a *parent element* D_p, see Fig. 1, (ii) the relations are "transformed" to $\hat{E} \in T^{(i)}_{h_i,\widehat{D}}$ using an affine-linear mapping and scaling arguments, (iii) by virtue of the mappings $\mathbf{\Phi}_i$ defined in (3.6) and relations (3.9), we express the results in every Ω_i.

Let D_p be the parent element e.g $[-x_b, x_b]^d \subset \mathbb{R}^d$, with diameter H_p, see Fig. 1. D_p is convex simply connected domain, thus for any $x \in \partial D_p, \exists x_0 \in D_p$ such that

$$(x - x_0) \cdot \mathbf{n}_{\partial D_p} > C_{D_p} \sim C_{H_p}. \tag{4.1}$$

Lemma 1. *For any $u \in W^{l,p}(D_p), l \geq 2, p > 1$ there is a $C := C_{H_p,d,p}$ such that the following trace inequality holds true*

$$\int_{\partial D_p} |u(s)|^p \, ds \leq C\big(\int_{D_p} |\nabla u(x)|^p \, dx + \int_{D_p} |u(x)|^p \, dx\big). \tag{4.2}$$

Proof. For $r = (x - x_0)$ we have

$$\int_{D_p} \nabla |u|^p \cdot r\, dx = \sum_{i=1}^{d} \int_{D_p} p|u|^{p-2} u \frac{\partial u}{\partial x_i} r_i\, dx = p \int_{D_p} |u|^{p-2} u \nabla u \cdot r\, dx. \tag{4.3}$$

The application of divergence theorem gives

$$\int_{D_p} \nabla |u|^p \cdot r\, dx = \int_{\partial D_p} |u|^p r \cdot \mathbf{n}_{\partial D_p}\, ds - \int_{D_p} |u|^p div(r)\, dx. \tag{4.4}$$

Hence, by (4.1), (4.3) and (4.4) it follows that

$$\int_{\partial D_p} |u|^p r \cdot \mathbf{n}_{\partial D_p}\, ds \;=\; p \int_{D_p} |u|^{p-2} u \nabla u \cdot r\, dx + \int_{D_p} |u|^p div(r)\, dx$$

and by (4.1), we get

$$C_{H_p} \int_{\partial D_p} |u|^p\, ds \leq p \int_{D_p} |u|^{p-2} u \nabla u \cdot r\, dx + \int_{D_p} |u|^p div(r)\, dx.$$

Applying Hölder and Youngs inequalities, we have

$$\begin{aligned} \int_{\partial D_p} |u|^p\, ds &\leq C_{H_p} \Big(C_{1,p} \big(\int_{D_p} |u|^p\, dx + |\nabla u|^p\, dx \big) + C_d \int_{D_p} |u|^p\, dx \Big) \\ &\leq C_{H_p,d,p} \Big(\int_{D_p} |u|^p\, dx + \int_{D_p} |\nabla u|^p\, dx \Big) \\ &= C_{H_p,d,p} \Big(\|u\|^p_{L^p(D_p)} + \|\nabla u\|^p_{L^p(D_p)} \Big). \end{aligned} \tag{4.5}$$

■

We point out that similar proof has been given in [24] in case of $p = 2$. D_p is considered as a reference element of any micro-element $\hat{E} \in T^{(i)}_{h_i,\widehat{D}}$ and let the linear affine map to be

$$\phi_{\hat{E}} : D_p \to \hat{E} \in T^{(i)}_{h_i,\widehat{D}}, \quad \phi_{\hat{E}}(x_{D_p}) = B x_{D_p} + b, \tag{4.6}$$

where $|det(B)| = |\hat{E}|$, see [34].

Using (4.6), we have $|u|_{W^{l,p}(D_p)} = h_{\hat{E}}^{l-\frac{d}{p}} |\hat{u}|_{W^{l,p}(\hat{E})}$ and applying the quasi uniformity of $T^{(i)}_{h_i,\widehat{D}}$, we derive by (4.2) that

$$h_{\hat{E}}^{-(d-1)} \int_{e \in \partial \hat{E}} |u|^p\, ds \;\leq\; C \Big(h_{\hat{E}}^{(0-\frac{d}{p})p} \int_{\hat{E}} |u|^p\, dx + h_{\hat{E}}^{p(1-\frac{d}{p})} \int_{\hat{E}} |\nabla u|^p\, dx \Big)$$

and directly we get

$$\int_{e\in\partial\hat{E}}|u|^p\,ds\leq C\Big(\frac{1}{h_i}\int_{\hat{E}}|u|^p\,dx+h_i^{p-1}\int_{\hat{E}}|\nabla u|^p\,dx\Big),\ \forall\hat{E}\in T^{(i)}_{h_i,\widehat{D}}. \quad (4.7)$$

Summing over all micro-elements $\hat{E}\in T^{(i)}_{h_i,\widehat{D}}$, we have

$$\int_{\hat{F}_i\in\partial\widehat{D}}|u|^p\,ds\leq C\Big(\frac{1}{h_i}\int_{\widehat{D}}|u|^p\,dx+h_i^{p-1}\int_{\widehat{D}}|\nabla u|^p\,dx\Big). \quad (4.8)$$

Finally, by making use of (3.9), we get the trace inequality expressed on every sub-domain

$$\int_{F_{ij}\in\mathcal{F}}|u|^p\,ds\leq C\Big(\frac{1}{h_i}\int_{\Omega_i}|u|^p\,dx+h_i^{p-1}\int_{\Omega_i}|\nabla u|^p\,dx\Big), \quad (4.9)$$

where the constant C is determined according to the C_m, C_M in (3.9).

Lemma 2. *(inverse estimates) For all $\phi_h\in\hat{\mathbb{B}}^{(i)}_{h_i}$ defined on $T^{(i)}_{h_i,\widehat{D}}$, there is a constant C depended on mesh quasi-uniformity parameters of the mesh but not on h_i, such that*

$$\|\nabla\phi_h\|^p_{L^p(\widehat{D})}\leq\frac{C}{h_i^p}\|\phi_h\|^p_{L^p(\widehat{D})} \quad (4.10)$$

Proof. The restriction of $\phi_h|_{\hat{E}}$ is a $B-$Spline polynomial of the same order. Considering the same (finite dimensional) space on the parent element D_p and by the equivalence norms on D_p we have, [34],

$$\|\nabla\phi_h\|^p_{L^p(D_p)}\leq C_{D_p}\|\phi_h\|^p_{L^p(D_p)}. \quad (4.11)$$

Applying scaling arguments and the mesh quasi-uniformity properties of $T^{(i)}_{h_i,\widehat{D}}$, the left and the right hand side of (4.11) can be expressed on every $\hat{E}\in T^{(i)}_{h_i,\widehat{D}}$ as

$$h_i^{p-\frac{d}{p}p}\|\nabla\phi_h\|^p_{L^p(\hat{E})}\leq Ch_i^{-\frac{d}{p}p}\|\phi_h\|^p_{L^p(\hat{E})}, \quad (4.12)$$

summing over all in (4.12) $\hat{E}\in T^{(i)}_{h_i,\widehat{D}}$, we can easily deduce (4.10).
■

Lemma 3. *(trace inequality on finite dimensional space) For all $\phi_h \in \hat{\mathbb{B}}^{(i)}_{h_i}$ defined on $T^{(i)}_{h_i,\widehat{D}}$ and for all $\hat{F}_i \in \partial\widehat{D}$, there is a constant C depended on mesh quasi-uniformity parameters of the mesh but not on h_i, such that*

$$\|\phi_h\|^p_{L^p(\hat{F}_i\in\partial\widehat{D})} \leq \frac{C}{h_i^p}\|\phi_h\|^p_{L^p(\widehat{D})} \tag{4.13}$$

Proof. *Applying the same scaling arguments as before and using the local quasi-uniformity of $T^{(i)}_{h_i,\widehat{D}}$, that is for every $\hat{e} \in \partial\hat{E}$ holds $|\hat{e}| \sim h_i$ we can show the following local trace inequality*

$$\|\phi_h\|^p_{L^p(\hat{e}\in\partial\hat{E})} \leq Ch_i^{-p}\|\phi_h\|^p_{L^p(\hat{E})} \tag{4.14}$$

summing over all $\hat{E} \in T^{(i)}_{h_i,\widehat{D}}$ that have an edge on $\hat{F}_i$ we deduce (4.13).
■

Next a Lemma for the relation among the $|\phi_h|_{W^{l,p}(\widehat{D})}$ and $|\phi_h|_{W^{m,p}(\widehat{D})}$.

Lemma 4. *Let $\phi_h \in \hat{\mathbb{B}}^{(i)}_{h_i}$ such that $\phi_h \in W^{l,p}(\hat{E})\cap W^{m,q}(\hat{E}), \forall \hat{E} \in T^{(i)}_{h_i,\widehat{D}}$, where $0 \leq m \leq l$, $1 \leq p,q \leq \infty$. Then there is a constant $C := C(l,p,m,q)$ depended on mesh quasi-uniformity parameters of the mesh but not on h_i, such that*

$$|\phi_h|_{W^{l,p}(\widehat{E})} \leq Ch_i^{m-l-\frac{d}{q}+\frac{d}{p}}|\phi_h|_{W^{m,q}(\widehat{E})} \tag{4.15}$$

Proof. *We mimic the analysis of Chp 4 in [34]. For any $\phi_h \in \hat{\mathbb{B}}^{(i)}_{h_i}|_{D_p}$, we have that*

$$|\phi_h|_{W^{l,p}(D_p)} \leq C|\phi_h|_{W^{m,q}(D_p)},\ \phi_h \in \hat{\mathbb{B}}^{(i)}_{h_i}|_{D_p}. \tag{4.16}$$

Using the scaling arguments as in proof of (4.7),

$$h_{\hat{E}}^{l-\frac{d}{p}}|\phi_h|_{W^{l,p}(\hat{E})} \leq Ch_{\hat{E}}^{m-\frac{d}{q}}|\phi_h|_{W^{m,q}(\hat{E})}$$

and it follows directly

$$|\phi_h|_{W^{l,p}(\hat{E})} \leq Ch_i^{m-l-\frac{d}{q}+\frac{d}{p}}|\phi_h|_{W^{m,q}(\hat{E})},\ \phi_h \in \hat{\mathbb{B}}^{(i)}_{h_i}. \tag{4.17}$$

For the particular case of $m = l = 0$ in (4.15), we have that

$$\|\phi_h\|_{L^p(\hat{E})} \leq Ch_i^{d(\frac{1}{p}-\frac{1}{q})}\|\phi_h\|_{L^q(\hat{E})}. \tag{4.18}$$

■

4.1 Analysis of the DGIGA discretization

Next, we study the convergence estimates of the method (3.16). We assume for the solution u that $u \in W^{l,2}_{\mathcal{S}} := W^{1,2}(\Omega) \cap W^{l,2}(\mathcal{S}(\Omega)),\ l \geq 2$. We consider the enlarged space $W^{l,2}_h := W^{l,2}_{\mathcal{S}} + \mathbb{B}_h(\mathcal{S}(\Omega))$, equipped with the broken DG-norm

$$\|u\|^2_{DG} = \sum_{i=1}^{N} \Big(\alpha^{(i)} \|\nabla u^{(i)}\|^2_{L^2(\Omega_i)} + p_i(u^{(i)}, u^{(i)})\Big),\ u \in W^{l,2}_h. \tag{4.19}$$

For the error analysis is necessary to show the continuity and coercivity properties of the bilinear form $a_h(.,.)$ of (3.16). Initially, we give a bound for the consistency terms.

Lemma 5. *For $(u, \phi_h) \in W^{l,2}_h \times \mathbb{B}_h(\mathcal{S}(\Omega))$, there are $C_{1,\varepsilon}, C_{2,\varepsilon} > 0$ such that for every $F_{ij} \in \mathcal{F}_I$*

$$\begin{aligned}|s_i| = \Big| \int_{F_{ij}} \{\alpha \nabla u\} \cdot \mathbf{n}_{F_{ij}} (\phi_h^{(i)} - \phi_h^{(j)})\, ds \Big| \leq \\ C_{1,\varepsilon}\Big(h_i \alpha^{(i)} \|\nabla u^{(i)}\|^2_{L^2(F_{ij})} + h_j \alpha^{(j)} \|\nabla u^{(j)}\|^2_{L^2(F_{ij})}\Big) + \\ \frac{1}{C_{2,\varepsilon}} \Big(\frac{\alpha^{(i)}}{h_i} + \frac{\alpha^{(j)}}{h_j} \Big) \|\phi_h^{(i)} - \phi_h^{(j)}\|^2_{L^2(F_{ij})}.\end{aligned} \tag{4.20}$$

Proof. Expanding the terms and applying Cauchy-Schwartz inequality yields

$$
\begin{aligned}
|s_i| \leq C\Big|\int_{F_{ij}} \{\alpha\nabla u\}\cdot \mathbf{n}_{F_{ij}}(\phi_h^{(i)}-\phi_h^{(j)})\,ds\Big| \leq \\
C\Big(\alpha^{(i)}\|\nabla u^{(i)}\|_{L^2(F_{ij})}+\alpha^{(j)}\|\nabla u^{(j)}\|_{L^2(F_{ij})}\Big)\|\phi_h^{(i)}-\phi_h^{(j)}\|_{L^2(F_{ij})}
\end{aligned}
$$

applying Young's inequality:

$$
\begin{aligned}
\alpha^{(i)}\|\nabla u^{(i)}\|_{L^2(F_{ij})}\|\phi_h^{(i)}-\phi_h^{(j)}\|_{L^2(F_{ij})} \leq C_{1,\varepsilon}h_i\alpha^{(i)}\|\nabla u^{(i)}\|^2_{L^2(F_{ij})}+ \\
\frac{\alpha^{(i)}}{C_{2,\varepsilon}h_i}\|\phi_h^{(i)}-\phi_h^{(j)}\|^2_{L^2(F_{ij})}
\end{aligned}
$$

we obtain

$$
\begin{aligned}
|s_i| \leq C_{1,\varepsilon}h_i\alpha^{(i)}\|\nabla u^{(i)}\|^2_{L^2(F_{ij})}+C_{1,\varepsilon}h_j\alpha^{(j)}\|\nabla u^{(j)}\|^2_{L^2(F_{ij})}+ \\
\frac{\alpha^{(i)}}{C_{2,\varepsilon}h_i}\|\phi_h^{(i)}-\phi_h^{(j)}\|^2_{L^2(F_{ij})}+\frac{\alpha^{(j)}}{C_{2,\varepsilon}h_j}\|\phi_h^{(i)}-\phi_h^{(j)}\|^2_{L^2(F_{ij})} = \\
C_{1,\varepsilon}\Big(h_i\alpha^{(i)}\|\nabla u^{(i)}\|^2_{L^2(F_{ij})}+h_j\alpha^{(j)}\|\nabla u^{(j)}\|^2_{L^2(F_{ij})}\Big)+ \\
\frac{1}{C_{2,\varepsilon}}\Big(\frac{\alpha^{(i)}}{h_i}+\frac{\alpha^{(j)}}{h_j}\Big)\|\phi_h^{(i)}-\phi_h^{(j)}\|^2_{L^2(F_{ij})}.
\end{aligned}
$$

■

Remark 1. In case where $F_i \in \mathcal{F}_B$, the corresponding bound can be derived by setting in (4.20) $\alpha^{(j)} = 0$ and $\phi_h^{(j)} = 0$.

Lemma 6. *(Discrete Coercivity) There exist a $C > 0$ independent of α and h_i, such that*

$$a_h(u_h,u_h) \geq C\|u_h\|^2_{DG},\ \ u_h \in \mathbb{B}_h(\mathcal{S}(\Omega)) \tag{4.21}$$

Proof. *By (3.16a), we have that*

$$a_h(u_h,u_h)=\sum_{i=1}^{N} a_i(u_h,u_h)-s_i(u_h,u_h)+p_i(u_h,u_h)=$$
$$\sum_{i=1}^{N}\alpha_i\|\nabla u_h\|^2_{L^2(\Omega_i)}-\sum_{F_{ij}\in\mathcal{F}}\frac{1}{2}\int_{F_{ij}}\{\alpha\nabla u_h\}\cdot\mathbf{n}_{F_{ij}}[\![u_h]\!]\,ds+$$
$$\sum_{F_{ij}\in\mathcal{F}}\mu\Big(\frac{\alpha^{(i)}}{h_i}+\frac{\alpha^{(j)}}{h_j}\Big)\|[\![u_h]\!]\|^2_{L^2(F_{ij})}. \qquad (4.22)$$

For the second term on the right hand side, Lemma 5 and the trace inequality (4.13) expressed on $F_{ij}\in\mathcal{F}$ yield the bound

$$-\sum_{F_{ij}\in\mathcal{F}}\frac{1}{2}\int_{F_{ij}}\{\alpha\nabla u_h\}\cdot\mathbf{n}_{F_{ij}}[\![u_h]\!]\,ds\geq$$
$$-C_{1,\varepsilon}\sum_{i=1}^{N}\alpha_i\|\nabla u_h\|^2_{L^2(\Omega_i)}-\sum_{F_{ij}\in\mathcal{F}}\frac{1}{C_{2,\varepsilon}}\Big(\frac{\alpha^{(i)}}{h_i}+\frac{\alpha^{(j)}}{h_j}\Big)\|[\![u_h]\!]\|^2_{L^2(F_{ij})}. \qquad (4.23)$$

Inserting (4.23) into (4.22) and choosing $C_{1,\varepsilon}<\frac{1}{2}$ and $\mu>\frac{2}{C_{2,\varepsilon}}$ we obtain (4.21).
■

Lemma 7. *(Boundedness) There are $C_1,C_2>0$ independent of h_i such that for all $(u,\phi_h)\in W_h^{l,2}\times\mathbb{B}_h(\mathcal{S}(\Omega))$*

$$a_h(u,\phi_h)\leq C_1\Big(\|u\|^2_{DG}+\sum_{F_{ij}\in\mathcal{F}}\alpha^{(i)}h_i\|\nabla u^{(i)}\|^2_{L^2(F_{ij})}\Big)+C_2\|\phi_h\|^2_{DG}. \qquad (4.24)$$

Proof. *We have by (3.16a) that*

$$a_h(u,\phi_h)=\sum_{i=1}^{N}\int_{\Omega_i}\alpha\nabla u\nabla\phi_h\,dx+\sum_{F_{ij}\in\mathcal{F}}\frac{1}{2}\int_{F_{ij}}\{\alpha\nabla u\}\cdot\mathbf{n}_{F_{ij}}[\![\phi_h]\!]\,ds+$$
$$\sum_{F_{ij}\in\mathcal{F}}\int_{F_{ij}}\Big(\frac{\mu\alpha^{(j)}}{h_j}+\frac{\mu\alpha^{(i)}}{h_i}\Big)[\![u]\!][\![\phi_h]\!]\,ds=T_1+T_2+T_3. \qquad (4.25)$$

Applying Cauchy-Schwartz inequality and consequently Young's inequality on every term in (4.25) yield the bounds

$$T_1 \leq C_1 \|u\|_{DG}^2 + C_2 \|\phi_h\|_{DG}^2,$$

for the term T_2, owing to the Lemma 5

$$\begin{aligned} T_2 \leq \sum_{F_{ij}\in\mathcal{F}} \Big(C_1 \alpha^{(i)} h_i \|\nabla u^{(i)}\|^2_{L^2(F_{ij})} + C_2 \Big(\frac{\mu\alpha^{(j)}}{h_j} + \frac{\mu\alpha^{(i)}}{h_i}\Big) \|[\![\phi_h]\!]\|^2_{L^2(F_{ij})} \Big) \\ \leq C_1 \sum_{F_{ij}\in\mathcal{F}} \alpha^{(i)} h_i \|\nabla u^{(i)}\|^2_{L^2(F_{ij})} + C_2 \|\phi_h\|^2_{DG}, \end{aligned}$$

$$\begin{aligned} T_3 \leq \sum_{F_{ij}\in\mathcal{F}} \Big(\frac{\mu\alpha^{(j)}}{h_j} + \frac{\mu\alpha^{(i)}}{h_i}\Big) \Big(C_1 \|[\![u]\!]\|^2_{L^2(F_{ij})} + C_2 \|[\![\phi_h]\!]\|^2_{L^2(F_{ij})} \Big) \\ \leq C_1 \|u\|^2_{DG} + C_2 \|\phi_h\|^2_{DG}. \end{aligned}$$

Substituting the bounds of T_1, T_2, T_3 into (4.25), we can derive (4.24).

■

In Chp 12 in [10], B-Spline intrpolants, say Π_h, are defined for $u \in L^p$ functions. Next, we consider the same interpolant $\Pi_h u$ and give an estimate of the interpolation error of $\Pi_h u$ to $u \in W^{l,2}(\Omega_i)$.

Lemma 8. *Let $m, l \geq 2$ be positive integers with $0 \leq m \leq l \leq k+1$ and let $E = \mathbf{\Phi}_i(\hat{E}), \hat{E} \in T^{(i)}_{h_i,\hat{D}}$. For $u \in W^{l,2}(\Omega_i)$ there exist an interpolant $\Pi_h u \in \mathbb{B}^{(i)}_h$ and a constant $C := C(\max_{l_0<l} \|D^{l_0}\mathbf{\Phi}_i\|^l_{L^\infty(E)})$ such that*

$$\sum_{E\in T^{(i)}_{h_i,\Omega_i}} |u - \Pi_h u|^2_{W^{m,2}(E)} \leq C h_i^{2(l-m)} \|u\|^2_{W^{l,2}(\Omega_i)} \tag{4.26}$$

Proof. The proof of (4.26) is included in Lemma 11 (see below) if we set $p = 2$. See also [9].

■

Next we give interpolation estimates on the interfaces and in $\|.\|_{DG}$-norm.

Lemma 9. *There exist constants $C_i := C(\max_{l_0<l} \|D^{l_0}\mathbf{\Phi}_i\|_{L^\infty(\Omega_i)}, \|u\|_{W^{l,2}(\Omega_i)})$ such that for all $F_{ij} \in \mathcal{F}$ the*

following estimates are true

$$h_i\alpha^{(i)}\|(\nabla u^{(i)} - \nabla \Pi_h u^{(i)}) \cdot \mathbf{n}_{F_{ij}}\|^2_{L^2(F_{ij})} \leq C_i h_i^{2l-2}, \tag{4.27a}$$

$$\Big(\frac{\alpha^{(j)}}{h_j} + \frac{\alpha^{(i)}}{h_i}\Big)\|u^{(i)} - \Pi_h u^{(i)}\|^2_{L^2(F_{ij})} \leq C_i\big(\alpha^{(i)}h_i^{2l-2} + \frac{\alpha^{(j)}h_i^{2l-1}}{h_j}\big), \tag{4.27b}$$

$$\|u - \Pi_h u\|^2_{DG} \leq \sum_{i=1}^{N} C_i\big(h_i^{2l-2} + \sum_{F_{ij}\in\mathcal{F}} \alpha^{(j)}\frac{h_i}{h_j}h_i^{2l-2}\big). \tag{4.27c}$$

Proof. For (4.27a): we apply the trace inequality (4.9) for $u := u^{(i)} - \Pi_h u^{(i)}$ *and consequently using the approximation estimate (4.26) the result easily follows.*
For (4.27b): we make use again of (4.9) and we get

$$\Big(\frac{\alpha^{(j)}}{h_j} + \frac{\alpha^{(i)}}{h_i}\Big)\|u^{(i)} - \Pi_h u^{(i)}\|^2_{L^2(F_{ij})} \leq$$
$$C_i\Big(\frac{\alpha^{(j)}}{h_j} + \frac{\alpha^{(i)}}{h_i}\Big)\Big(\frac{1}{h_i}\|u^{(i)} - \Pi_h u^{(i)}\|^2_{L^2(\Omega_i)} + h_i\|\nabla u^{(i)} - \nabla\Pi_h u^{(i)}\|^2_{L^2(\Omega_i)} \leq$$
$$C_i\Big(\frac{\alpha^{(j)}}{h_j} + \frac{\alpha^{(i)}}{h_i}\Big)h_i^{2l-1} \leq C_i\big(\alpha^{(i)}h_i^{2l-2} + \frac{\alpha^{(j)}h_i^{2l-1}}{h_j}\big)$$

Recalling the approximation result (4.26) and using (4.27b) we can deduce estimate (4.27c).
■

In order to proceed and to give an estimate for the error $\|u - u_h\|_{DG}$, we need to show that the weak solution satisfies the form (3.16a).

Lemma 10. *(Consistency of the weak solution.) The solution u satisfies the variational formulation (3.16),*

$$\sum_{i=1}^{N}\Big(\int_{\Omega_i}\alpha\nabla u\cdot\nabla\phi_h\,dx - \sum_{F_{ij}\in\mathcal{F}_I}\Big(\int_{F_{ij}}\{\alpha\nabla u\}\cdot\mathbf{n}_{F_{ij}}[\![\phi_h]\!]\,ds +$$
$$\Big(\frac{\mu\alpha^{(i)}}{h_i} + \frac{\mu\alpha^{(j)}}{h_j}\Big)\int_{F_{ij}}[\![u]\!][\![\phi_h]\!]\,ds\Big) +$$
$$\sum_{F_i\in\mathcal{F}_B}\Big(\int_{F_i}\alpha^{(i)}\nabla u\cdot\mathbf{n}_{F_i}\phi_h\,ds + \frac{\mu\alpha^{(i)}}{h_i}\int_{F_i}u\phi_h\,ds\Big) =$$
$$\sum_{i=1}^{N}\int_{\Omega_i}f\phi_h\,dx + \sum_{F_i\in\mathcal{F}_B}\frac{\mu\alpha^{(i)}}{h_i}\int_{F_i}u_D\phi_h\,ds. \tag{4.28}$$

Proof. We multiply (2.1) by $\phi_h \in \mathbb{B}_h(\mathcal{S}(\Omega))$ and integrating by parts on each sub-domain Ω_i we get

$$\int_{\Omega_i} \alpha \nabla u \cdot \nabla \phi_h \, dx - \int_{\partial\Omega_i} \alpha \nabla u \cdot \mathbf{n}_{\partial\Omega_i} \phi_h \, ds = \int_{\Omega_i} f \phi_h \, dx.$$

Summing over all sub-domains

$$\sum_{i=1}^{N} \int_{\Omega_i} \alpha \nabla u \cdot \nabla \phi_h \, dx - \sum_{F_{ij} \in \mathcal{F}} \int_{F_{ij}} [\![\alpha \nabla u \phi_h]\!] \cdot \mathbf{n}_{F_{ij}} \, ds = \sum_{i=1}^{N} \int_{\Omega_i} f \phi_h \, dx. \tag{4.29}$$

The regularity assumption $u \in W^{l,2}_{\mathcal{S}}$ implies that $[\![\alpha \nabla u]\!] \cdot \mathbf{n}_{F_{ij}} = 0$. Making use of the identity

$$[\![ab]\!] = a_1 b_1 - a_2 b_2 = \{a\}[\![b]\!] + [\![a]\!]\{b\},$$

relation (4.29) can be reformulated as

$$\sum_{i=1}^{N} \int_{\Omega_i} \alpha \nabla u \cdot \nabla \phi_h \, dx - \sum_{F_{ij} \in \mathcal{F}_I} \frac{1}{2} \int_{F_{ij}} \{\alpha \nabla u\} \cdot \mathbf{n}_{F_{ij}} [\![\phi_h]\!] \, ds + \sum_{F_i \in \mathcal{F}_B} \int_{F_i} \alpha \nabla u \cdot \mathbf{n}_{F_i} \phi_h \, ds = \int_{\Omega_i} f \phi_h \, dx. \tag{4.30}$$

The continuity of u implies further that

$$\sum_{F_{ij} \in \mathcal{F}_I} \Big(\frac{\mu \alpha^{(i)}}{h_i} + \frac{\mu \alpha^{(j)}}{h_j} \Big) \int_{F_{ij}} [\![u]\!][\![\phi_h]\!] \, ds + \sum_{F_i \in \mathcal{F}_B} \frac{\mu \alpha^{(i)}}{h_i} \int_{F_i} u \phi_h \, ds = \sum_{F_i \in \mathcal{F}_B} \frac{\mu \alpha^{(i)}}{h_i} \int_{F_i} u_D \phi_h \, ds. \tag{4.31}$$

Finally, adding the terms of (4.31) and (4.30) we can deduce (4.28). ■

We can now give an error estimate in $\|.\|_{DG}$-norm.

Theorem 1. *Let $u \in W^{l,2}_{\mathcal{S}}$ solves (2.2) and let $u_h \in \mathbb{B}_h(\mathcal{S}(\Omega))$ solves the discrete problem (3.16). Then the error $u - u_h$ satisfies*

$$\|u - u_h\|^2_{DG} < \sum_{i=1}^{N} C_i \Big(h_i^{2l-2} + \sum_{F_{ij} \in \mathcal{F}} \alpha^{(j)} \frac{h_i}{h_j} h_i^{2l-2} \Big), \tag{4.32}$$

where $C_i := C(\max_{l_0 < l} \|D^{l_0} \mathbf{\Phi}_i\|^l_{L^\infty(\Omega_i)}, \|u\|_{W^{l,2}(\Omega_i)})$.

Proof. *Let $\mathit{\Pi}_h u \in \mathbb{B}_h(\mathcal{S}(\Omega))$ as in Lemma 8, by subtracting (4.28) from (3.16a) we get*

$$a_h(u_h, \phi_h) = a_h(u, \phi_h),$$

and adding $-a_h(\mathit{\Pi}_h u, \phi_h)$ on both sides

$$a_h(u_h - \mathit{\Pi}_h u, \phi_h) = a_h(u - \mathit{\Pi}_h u, \phi_h). \tag{4.33}$$

Note that $u_h - \mathit{\Pi}_h u \in \mathbb{B}_h(\mathcal{S}(\Omega))$. Therefore we may set $\phi_h = u_h - \mathit{\Pi}_h u$ in (4.33), and consequently applying Lemma 6 and Lemma 7 we find

$$\|u_h - \mathit{\Pi}_h u\|^2_{DG} \leq C\Big(\|u - \mathit{\Pi}_h u\|^2_{DG} + \sum_{F_{ij}\in\mathcal{F}} \alpha^{(i)} h_i \|\nabla(u^{(i)} - \mathit{\Pi}_h u^{(i)})\|^2_{L^2(F_{ij})}\Big) \tag{4.34}$$

Using the triangle inequality

$$\|u - u_h\|^2_{DG} \leq \|u_h - \mathit{\Pi}_h u\|^2_{DG} + \|u - \mathit{\Pi}_h u\|^2_{DG} \tag{4.35}$$

in (4.34) and consequently applying the estimates of (4.27) we can obtain (4.32).
■

5 Low-Regularity solutions

In this section, we investigate the convergence of the discrete solution u_h produced by the proposed DGIGA method (3.16), under the assumption that the weak solution u of the model problem (2.1) has less regularity, that is $u \in W^{l,p}_{\mathcal{S}} := W^{1,2}(\Omega) \cap W^{l,p}(\mathcal{S}(\Omega)),\ l \geq 2, p \in (\frac{2d}{d+2(l-1)}, 2]$. Problems with low regularity solutions we can be found in several cases, as for example, when the domains have singular boundary points, points with changing boundary conditions, see e.g. [35], [36]. We use the enlarged space $W^{l,p}_h = W^{l,p}_{\mathcal{S}} + \mathbb{B}_h(\mathcal{S}(\Omega))$ and will show that the DGIGA method converges in optimal rate with respect to $\|.\|_{DG}$ norm defined in (4.19). We develop our analysis inspired by the techniques used in [30], [19]. A basic tool that we will use is the Sobolev embeddings theorems, see [32],[31]. Let $l = j + m \geq 2$, then for $j = 0$ or $j = 1$ it holds that

$$\|u\|_{W^{j,2}(\Omega_i)} \leq C(l, p, 2, \Omega_i)\|u\|_{W^{l,p}(\Omega_i)}, \text{ for } p > \frac{2d}{d + 2m}. \tag{5.1}$$

We start by proving an approximation estimate for interpolants of the tensor product splines of degree k defined in (3.6).

Lemma 11. *(Interpolation estimates). Let* $u \in W^{l,p}(\Omega_i)$ *with* $l \geq 2, p \in (\max\{1, \frac{2d}{d+2(l-1)}\}, 2]$ *and let* $E = \mathbf{\Phi}_i(\hat{E}), \hat{E} \in T^{(i)}_{h_i,\hat{D}}$. *Then for* $0 \leq m \leq l \leq k+1$, *there exist an interpolant* $\Pi_h u \in \mathbb{B}_h(\mathcal{S}(\Omega))$ *and constants* $C_i := C_i\big(\max_{l_0 \leq l} \|D^{l_0}\mathbf{\Phi}_i\|_{L^\infty(\Omega_i)}), \|u\|_{W^{l,p}(\Omega_i)}\big)$, *such that*

$$\sum_{E \in T^{(i)}_{h_i,\Omega_i}} |u - \Pi_h u|^p_{W^{m,p}(E)} \leq h_i^{p(l-m)} C_i. \tag{5.2}$$

Furthermore for the interface terms and the $\|.\|_{DG}$ *norm we have the estimates*

- $h_i^\beta \|\nabla u^{(i)} - \nabla \Pi_h u^{(i)}\|^p_{L^p(F_{ij})} \leq C_i C_{d,p} h_i^{p(l-1)-1+\beta}, \tag{5.3a}$
- $$\Big(\frac{\alpha^{(j)}}{h_j} + \frac{\alpha^{(i)}}{h_i}\Big)\|[\![u - \Pi_h u]\!]\|^2_{L^2(F_{ij})} \leq \tag{5.3b}$$
  $$C_i \alpha^{(j)} \frac{h_i}{h_j}\Big(h_i^{\delta(p,d)}\|u\|^p_{W^{l,p}(\Omega_i)}\Big)^2 + C_j \alpha^{(i)} \frac{h_j}{h_i}\Big(h_j^{\delta(p,d)}\|u\|_{W^{l,p}(\Omega_j)}\Big)^2 +$$
  $$C_j\Big(h_j^{\delta(p,d)}\|u\|_{W^{l,p}(\Omega_j)}\Big)^2 + C_i\Big(h_i^{\delta(p,d)}\|u\|_{W^{l,p}(\Omega_i)}\Big)^2,$$
- $$\|u - \Pi_h u\|^2_{DG} \leq \sum_{i=1}^N C_i\Big(h_i^{\delta(p,d)}\|u\|_{W^{l,p}(\Omega_i)}\Big)^2 + \tag{5.3c}$$
  $$\sum_{F_{ij}\in\mathcal{F}} C_i \alpha^{(j)} \frac{h_i}{h_j}\Big(h_i^{\delta(p,d)}\|u\|_{W^{l,p}(\Omega_i)}\Big)^2$$

where $\delta(p,d) = l + (\frac{d}{2} - \frac{d}{p} - 1)$.

Proof. We give the proof of (5.2) based on the results of Chap 12 and Chap 13 in [10]. For $f \in W^{l,p}(\hat{D})$, there exist a tensor-product polynomial of order m, $T^m f$ such that for every $\hat{E} \in T^{(i)}_{h_i,\hat{D}}$ holds, [34],

$$|f - T^m f|_{W^{m,p}(\hat{E})} \leq C_{d,l,m} h_i^{l-m} |f|_{W^{l,p}(D^{(i)}_{\hat{E}})}. \tag{5.4}$$

Because of $m \leq k$ is $\Pi_h(T^m f) = T^m f$ and it also holds that $\|\Pi_h f\|_{L^p(\hat{E})} \leq C\|f\|_{L^p(D_{\hat{E}}^{(i)})}$. Using these results, we have that

$$\begin{aligned}|u-\Pi_h u|_{W^{m,p}(\hat{E})} &\leq |u-T^m u|_{W^{m,p}(\hat{E})} + |\Pi_h u - T^m u|_{W^{m,p}(\hat{E})} \\ &\leq |u-T^m u|_{W^{m,p}(\hat{E})} + |\Pi_h(u-T^m u)|_{W^{m,p}(\hat{E})} \\ &\leq C_1 h_i^{l-m}|u|_{W^{l,p}(D_{\hat{E}}^{(i)})} + C_2 h_i^{-m+\frac{d}{p}-\frac{d}{p}}|\Pi_h(u-T^m u)|_{L^p(\hat{E})} \text{ (by (4.10))} \\ &\leq C_1 h_i^{l-m}|u|_{W^{l,p}(D_{\hat{E}}^{(i)})} + C_2 h_i^{-m}|u-T^m u|_{L^p(\hat{E})} \text{ (by (5.4))} \\ &\leq C h_i^{l-m}|u|_{W^{l,p}(D_{\hat{E}}^{(i)})}. \end{aligned} \tag{5.5}$$

Recalling (3.9), the above inequality is expressed on every $E \in T_{h_i,\Omega_i}^{(i)}$. Then, taking the $p-th$ power and summing over the elements we obtain the interpolation estimate (5.2).

For the proof of (5.3a): applying (4.9) and using the uniformity of the mesh we get

$$\begin{aligned} h_i^{\beta}\|\nabla u^{(i)} - \nabla\Pi_h u^{(i)}\|^p_{L^p(F_{ij})} &\leq C_i C_{d,p} h_i^{\beta}\big(\frac{1}{h_i}\|\nabla u^{(i)} - \nabla \Pi_h u^{(i)}\|^p_{L^p(\Omega_i)} + \\ & h_i^{p-1}\|\nabla^2 u^{(i)} - \nabla^2 \Pi_h u^{(i)}\|^p_{L^p(\Omega_i)}\big) \leq^{by\ (5.2)} C_i C_{d,p} h_i^{p(l-1)-1+\beta}. \end{aligned} \tag{5.6}$$

For the proof of (5.3b): we first make use of the trace inequality (4.9)

$$\begin{aligned} \frac{\alpha^{(i)}}{h_i}\|u^{(i)} - \Pi_h u^{(i)}\|^2_{L^2(F_{ij})} &\leq C_i C_{d,p}\alpha^{(i)}\big(\frac{1}{h_i^2}\int_{\Omega_i}|u^{(i)} - \Pi_h u^{(i)}|^2\,dx \\ &+ \int_{\Omega_i}|\nabla(u^{(i)} - \Pi_h u^{(i)})|^2\,dx\big) = \\ C_i C_{d,p}\alpha^{(i)}\Big(\frac{1}{h_i^2}\sum_{E\in T_{h_i,\Omega_i}^{(i)}}\int_E |u^{(i)} - \Pi_h u^{(i)}|^2\,dx &+ \sum_{E\in T_{h_i,\Omega_i}^{(i)}}\int_E|\nabla(u^{(i)} - \Pi_h u^{(i)})|^2\,dx\Big). \end{aligned} \tag{5.7}$$

The Sobolev embedding (5.1) gives

$$\|u\|_{L^2(D_p)} \leq C(p,2,D_p)\big(\|u\|^p_{L^p(D_p)} + |u|^p_{W^{1,p}(D_p)}\big)^{\frac{1}{p}}. \tag{5.8}$$

Using the scaling arguments, see (4.6), and the bounds (3.9) we can derive the coresponding expression of (5.8) on every $E \in T^{(i)}_{h_i,\Omega_i}$,

$$h_i^{\frac{-d}{2}} \|u\|_{L^2(E)} \leq C_i h_i^{\frac{-d}{p}} \left(\|u\|^p_{L^p(E)} + h_i^p |u|^p_{W^{1,p}(E)}\right)^{\frac{1}{p}}$$

and a straight forward computation gives

$$h_i^{-2}\|u\|^2_{L^2(E)} \leq C_i h_i^{2(\frac{d}{2}-\frac{d}{p}-1)} \left(\|u\|^p_{L^p(E)} + h_i^p|u|^p_{W^{1,p}(E)}\right)^{\frac{2}{p}}. \quad (5.9)$$

Proceeding in the same manner, we can obtain

$$\|u\|^2_{W^{1,2}(E)} \leq C_i h_i^{2(\frac{d}{2}-\frac{d}{p}-1)} \left(\|u\|^p_{L^p(E)} + h_i^p|u|^p_{W^{1,p}(E)} + h_i^{2p}|u|^p_{W^{2,p}(E)}\right)^{\frac{2}{p}}. \quad (5.10)$$

Setting in (5.9) and (5.10) $u := u^{(i)} - \Pi_h u^{(i)}$ and applying the approximation result (5.2), we can show that

$$\begin{aligned}
&\sum_{E\in T^{(i)}_{h_i,\Omega_i}} \alpha^{(i)}\left(h_i^{-2}\|u^{(i)} - \Pi_h u^{(i)}\|^2_{L^2(E)} + \|u^{(i)} - \Pi_h u^{(i)}\|^2_{W^{1,2}(E)}\right) \\
&\leq \sum_{E\in T^{(i)}_{h_i,\Omega_i}} \left(\alpha^{(i)} C_i h_i^{l+(\frac{d}{2}-\frac{d}{p}-1)} \|u\|_{W^{l,p}(D^{(i)}_E)}\right)^2 \leq \ \textit{(note that } f(x) = (a^x+b^x)^{\frac{1}{x}} \downarrow\textit{)} \\
&\alpha^{(i)} C_i \Big(\sum_{E\in T^{(i)}_{h_i,\Omega_i}} \left(h_i^{lp+p(\frac{d}{2}-\frac{d}{p}-1)} \|u\|^p_{W^{l,p}(D^{(i)}_E)}\right)\Big)^{\frac{2}{p}} \leq \alpha^{(i)} C_i \left(h_i^{l+(\frac{d}{2}-\frac{d}{p}-1)} \|u\|_{W^{l,p}(\Omega_i)}\right)^2.
\end{aligned} \quad (5.11)$$

Moreover, (5.11) implies that

$$\frac{\alpha^{(j)} h_i}{h_j} \frac{1}{h_i} \|u^{(i)} - \Pi_h u^{(i)}\|^2_{L^2(F_{ij})} \leq C_i \frac{\alpha^{(j)} h_i}{h_j} \left(h_i^{l+(\frac{d}{2}-\frac{d}{p}-1)} \|u\|_{W^{l,p}(\Omega_i)}\right)^2, \quad (5.12)$$

similarly

$$\frac{\alpha^{(i)} h_j}{h_i} \frac{1}{h_j} \|u^{(j)} - \Pi_h u^{(j)}\|^2_{L^2(F_{ji})} \leq C_i \frac{\alpha^{(i)} h_j}{h_i} \left(h_j^{l+(\frac{d}{2}-\frac{d}{p}-1)} \|u\|_{W^{l,p}(\Omega_j)}\right)^2. \quad (5.13)$$

Now, returning to the inequality (5.3b) and using (5.11),(5.12) and (5.13), we find

$$
\begin{aligned}
&\Big(\frac{\alpha^{(j)}}{h_j}+\frac{\alpha^{(i)}}{h_i}\Big)\|[\![u-\Pi_h u]\!]\|^2_{L^2(F_{ij})}\leq\\
&\quad\frac{\alpha^{(j)}h_i}{h_j}\frac{1}{h_i}\|u^{(i)}-\Pi_h u^{(i)}\|^2_{L^2(F_{ij})}+\frac{\alpha^{(i)}h_j}{h_i}\frac{1}{h_j}\|u^{(j)}-\Pi_h u^{(j)}\|^2_{L^2(F_{ji})}\\
&\qquad+\frac{\alpha^{(j)}}{h_j}\|u^{(j)}-\Pi_h u^{(j)}\|^2_{L^2(F_{ji})}+\frac{\alpha^{(i)}}{h_i}\|u^{(i)}-\Pi_h u^{(i)}\|^2_{L^2(F_{ij})}\\
&\leq C_i\frac{\alpha^{(j)}h_i}{h_j}\Big(h_i^{l+(\frac{d}{2}-\frac{d}{p}-1)}\|u\|_{W^{l,p}(\Omega_i)}\Big)^2+C_j\frac{\alpha^{(i)}h_j}{h_i}\Big(h_j^{l+(\frac{d}{2}-\frac{d}{p}-1)}\|u\|_{W^{l,p}(\Omega_j)}\Big)^2\\
&\quad+C_j\Big(h_j^{l+(\frac{d}{2}-\frac{d}{p}-1)}\|u\|_{W^{l,p}(\Omega_j)}\Big)^2+C_i\Big(h_i^{l+(\frac{d}{2}-\frac{d}{p}-1)}\|u\|_{W^{l,p}(\Omega_i)}\Big)^2.
\end{aligned}
\tag{5.14}
$$

For the proof (5.3c), we recall the definition (4.19) for $u-\Pi_h u$ and have

$$
\begin{aligned}
\|u-\Pi_h u\|^2_{DG}=\sum_{i=1}^{N}\Big(&\alpha^{(i)}\|\nabla(u^{(i)}-\Pi_h u^{(i)})\|^2_{L^2(\Omega_i)}\\
&+\sum_{F_{ij}\in\mathcal{F}}\Big(\frac{\mu\alpha^{(i)}}{h_i}+\frac{\mu\alpha^{(j)}}{h_j}\Big)\|[\![u-\Pi_h u]\!]\|^2_{L^2(F_{ij})}\Big).
\end{aligned}
\tag{5.15}
$$

Estimating the first term on the right hand side in (5.15) by (5.11) and the second term by (5.14), the approximation estimate (5.3c) follows.
■

We need further discrete coercivity, consistency and boundedness. The discrete coercivity (Lemma 6) holds true for this low regularity case, too. Using the same arguments as in Lemma 10, we can prove the consistency for u. Due to assumed regularity of the solution, the normal interface fluxe $(\alpha\nabla u)|_{\Omega_i}\cdot\mathbf{n}_{F_{ij}}$ belongs (in general) to $L^p(F_{ij})$. Thus, we need to prove the boundedness for $a_h(.,.)$ by estimating the flux terms (3.16d) in different way than this in Lemma 7. We work in a similar way as in [26] and prove the following bound for the interface fluxes.

Lemma 12. *There is a constant $C := C(p,2)$ such that the following inequality for $(u,\phi_h) \in W_h^{l,p} \times \mathbb{B}_h(\mathcal{S}(\Omega))$ holds true*

$$\sum_{F_{ij}\in\mathcal{F}} \frac{1}{2}\int_{F_{ij}} \{\alpha\nabla u\}\cdot \mathbf{n}_{F_{ij}} [\![\phi_h]\!]\, ds \leq \tag{5.16}$$
$$C\Big(\sum_{F_{ij}\in\mathcal{F}} \alpha^{(i)} h_i^{1+\gamma_{p,d}} \|\nabla u^{(i)}\|^p_{L^p(F_{ij})} + \alpha^{(j)} h_j^{1+\gamma_{p,d}} \|\nabla u^{(j)}\|^p_{L^p(F_{ij})}\Big)^{\frac{1}{p}} \|\phi_h\|_{DG},$$

where $\gamma_{p,d} = \dfrac{1}{2}d(p-2)$.

Proof. For the interface edge $e_{ij} \subset F_{ij}$ Hölder inequality yield

$$\frac{1}{2}\int_{e_{ij}} \frac{1}{2}|\alpha^{(i)}\nabla u^{(i)} + \alpha^{(j)}\nabla u^{(j)}||[\![\phi_h]\!]|\, ds \leq$$
$$C\int_{e_{ij}} (\alpha^{(i)} h_i^{1+\gamma_{p,d}})^{\frac{1}{p}} |\nabla u^{(i)}| \frac{{\alpha^{(i)}}^{\frac{1}{q}}}{h_i^{\frac{1+\gamma_{p,d}}{p}}} |[\![\phi_h]\!]| + (\alpha^{(j)} h_j^{1+\gamma_{p,d}})^{\frac{1}{p}} |\nabla u^{(j)}| \frac{{\alpha^{(j)}}^{\frac{1}{q}}}{h_j^{\frac{1+\gamma_{p,d}}{p}}} |[\![\phi_h]\!]|\, ds$$
$$\leq C(\alpha^{(i)} h_i^{1+\gamma_{p,d}})^{\frac{1}{p}} \|\nabla u^{(i)}\|_{L^p(e_{ij})} \frac{{\alpha^{(i)}}^{\frac{1}{q}}}{h_i^{\frac{1+\gamma_{p,d}}{p}}} \|[\![\phi_h]\!]\|_{L^q(e_{ij})}$$
$$+ C(\alpha^{(j)} h_j^{1+\gamma_{p,d}})^{\frac{1}{p}} \|\nabla u^{(j)}\|_{L^p(e_{ij})} \frac{{\alpha^{(j)}}^{\frac{1}{q}}}{h_j^{\frac{1+\gamma_{p,d}}{p}}} \|[\![\phi_h]\!]\|_{L^q(e_{ij})}. \tag{5.17}$$

We employ the inverse inequality (4.18) with $p = q > 2$, $q = 2$ and use the analytical form $\frac{1+\gamma_{p,d}}{p} = \frac{2+d(p-2)}{2p}$ to express the jump terms in (5.17) in the convenient L^2 form as follows

$$\frac{{\alpha^{(i)}}^{\frac{1}{q}}}{h_i^{\frac{2+d(p-2)}{2p}}} \|[\![\phi_h]\!]\|_{L^q(e_{ij})} \leq C_{inv,p,2} {\alpha^{(i)}}^{\frac{1}{q}} h_i^{(d-1)(\frac{1}{q}-\frac{1}{2}) - \frac{2+d(p-2)}{2p}} \|[\![\phi_h]\!]\|_{L^2(e_{ij})}$$
$$\leq C_{inv,p,2} {\alpha^{(i)}}^{\frac{1}{q}} h_i^{\frac{-1}{2}} \|[\![\phi_h]\!]\|_{L^2(e_{ij})}. \tag{5.18}$$

Inserting the result (5.18) into (5.17) and summing over all $e_{ij} \in F_{ij}$ we obtain for $q > 2$,

$$
\begin{aligned}
&\frac{1}{2}\int_{F_{ij}} \{\alpha\nabla u\}\cdot \mathbf{n}_{F_{ij}} [\![\phi_h]\!]\, ds \leq C \sum_{e_{ij}\in F_{ij}} \int_{e_{ij}} |\alpha^{(i)}\nabla u^{(i)} + \alpha^{(j)}\nabla u^{(j)}||[\![\phi_h]\!]|\, ds \\
&\leq C\Big(\sum_{e_{ij}\in F_{ij}} \alpha^{(i)} h_i^{1+\gamma_{p,d}} \|\nabla u^{(i)}\|^p_{L^p(e_{ij})}\Big)^{\frac{1}{p}} \Big(\sum_{e_{ij}\in F_{ij}} \alpha^{(i)} \big(\frac{1}{h_i^{\frac{1}{2}}}\|[\![\phi_h]\!]\|_{L^2(e_{ij})}\big)^q\Big)^{\frac{1}{q}} \\
&+C\Big(\sum_{e_{ij}\in F_{ij}} \alpha^{(j)} h_j^{1+\gamma_{p,d}} \|\nabla u^{(j)}\|^p_{L^p(e_{ij})}\Big)^{\frac{1}{p}} \Big(\sum_{e_{ij}\in F_{ij}} \alpha^{(j)} \big(\frac{1}{h_j^{\frac{1}{2}}}\|[\![\phi_h]\!]\|_{L^2(e_{ij})}\big)^q\Big)^{\frac{1}{q}}.
\end{aligned}
\tag{5.19}
$$

Now, using that the function $f(x) = (\lambda\alpha^x + \lambda\beta^x)^{\frac{1}{x}}$, $\lambda > 0, x > 2$ is decreasing, we estimate the "q-power terms" in the sum of the right hand side in (5.19) as follows

$$
\begin{aligned}
\Big(\sum_{e_{ij}\in F_{ij}} \alpha^{(j)} \big(\frac{1}{h_j^{\frac{1}{2}}}\|[\![\phi_h]\!]\|_{L^2(e_{ij})}\big)^q\Big)^{\frac{1}{q}} &\leq \Big(\sum_{e_{ij}\in F_{ij}} \alpha^{(j)} \big(\frac{1}{h_j^{\frac{1}{2}}}\|[\![\phi_h]\!]\|_{L^2(e_{ij})}\big)^2\Big)^{\frac{1}{2}} \\
&\leq \Big(\big(\frac{\mu\alpha^{(i)}}{h_i} + \frac{\mu\alpha^{(j)}}{h_j}\big)\|[\![\phi_h]\!]\|^2_{L^2(F_{ij})}\Big)^{\frac{1}{2}},
\end{aligned}
\tag{5.20}
$$

and thus applying (5.20) into (5.19) we get

$$
\begin{aligned}
&\frac{1}{2}\int_{F_{ij}} \{\alpha\nabla u\}\cdot \mathbf{n}_{F_{ij}} [\![\phi_h]\!]\, ds \leq \\
&\qquad 2C\Big(\alpha^{(i)} h_i^{1+\gamma_{p,d}} \|\nabla u^{(i)}\|^p_{L^p(F_{ij})} + \alpha^{(j)} h_j^{1+\gamma_{p,d}} \|\nabla u^{(j)}\|^p_{L^p(F_{ij})}\Big)^{\frac{1}{p}} \\
&\qquad\qquad \Big(\big(\frac{\mu\alpha^{(i)}}{h_i} + \frac{\mu\alpha^{(j)}}{h_j}\big)\|[\![\phi_h]\!]\|^2_{L^2(F_{ij})}\Big)^{\frac{1}{2}}.
\end{aligned}
\tag{5.21}
$$

We sum over all $F_{ij} \in \mathcal{F}$ in (5.21) and consequently we apply Hölder inequality

$$
\begin{aligned}
&\frac{1}{2}\sum_{F_{ij}\in\mathcal{F}}\int_{F_{ij}} \{\alpha\nabla u^{(i)}\} [\![\phi_h]\!]\, ds \leq \\
&\quad 2C\Big(\sum_{F_{ij}\in\mathcal{F}} \alpha^{(i)} h_i^{1+\gamma_{p,d}} \|\nabla u^{(i)}\|^p_{L^p(F_{ij})} + \alpha^{(j)} h_j^{1+\gamma_{p,d}} \|\nabla u^{(j)}\|^p_{L^p(F_{ij})}\Big)^{\frac{1}{p}} \\
&\qquad\qquad \Big(\sum_{F_{ij}\in\mathcal{F}}\Big(\big(\frac{\mu\alpha^{(i)}}{h_i} + \frac{\mu\alpha^{(j)}}{h_j}\big)\|[\![\phi_h]\!]\|^2_{L^2(F_{ij})}\Big)^{\frac{q}{2}}\Big)^{\frac{1}{q}}.
\end{aligned}
\tag{5.22}
$$

Following in much the same arguments as in proof of (5.20), we can bound the second $\sum_{F_{ij}}$ *in (5.22) as*

$$\Big(\sum_{F_{ij}\in\mathcal{F}}\Big(\big(\frac{\mu\alpha^{(i)}}{h_i}+\frac{\mu\alpha^{(j)}}{h_j}\big)\|[\![\phi_h]\!]\|^2_{L^2(F_{ij})}\Big)^{\frac{q}{2}}\Big)^{\frac{1}{q}}\leq$$
$$\Big(\sum_{F_{ij}\in\mathcal{F}}\big(\frac{\mu\alpha^{(i)}}{h_i}+\frac{\mu\alpha^{(j)}}{h_j}\big)\|[\![\phi_h]\!]\|^2_{L^2(F_{ij})}\Big)^{\frac{1}{2}}\leq\|\phi_h\|_{DG}. \quad (5.23)$$

Using (5.22) and (5.23), we can easily obtain (5.16).
■

Lemma 13. *(boundedness) There is a* $C := C_{p,2}$ *independent of* h_i *such that* $\forall (u,\phi_h)\in W_h^{l,p}\times\mathbb{B}_h(\mathcal{S}(\Omega))$

$$a_h(u,\phi_h)\leq C(\|u\|^p_{DG}+\sum_{F_{ij}\in\mathcal{F}}h_i^{1+\gamma_{p,d}}\alpha^{(i)}\|\nabla u^{(i)}\|^p_{L^p(F_{ij})}+ \quad (5.24)$$
$$h_j^{1+\gamma_{p,d}}\alpha^{(j)}\|\nabla u^{(j)}\|^p_{L^p(F_{ij})}\Big)^{\frac{1}{p}}\|\phi_h\|_{DG},$$

Proof. We estimate the terms of $a_h(u,\phi_h)$ *in (3.16b) separately. Applying Cauchy-Schwartz for the terms (3.16c) and (3.16e) we have*

$$\sum_{i=1}^{N}a_i(u,\phi_h)\leq C\|u\|_{DG}\|\phi_h\|_{DG} \quad (5.25a)$$
$$\sum_{i=1}^{N}p_i(u,\phi_h)\leq C\|u\|_{DG}\|\phi_h\|_{DG}. \quad (5.25b)$$

For the term (3.16d) we use Lemma 12

$$\sum_{i=1}^{N}s_i(u,\phi_h)\leq C\Big(\sum_{F_{ij}\in\mathcal{F}}\alpha^{(i)}h_i^{1+\gamma_{p,d}}\|\nabla u^{(i)}\|^p_{L^p(F_{ij})}+$$
$$\alpha^{(j)}h_j^{1+\gamma_{p,d}}\|\nabla u^{(j)}\|^p_{L^p(F_{ij})}\Big)^{\frac{1}{p}}\|\phi_h\|_{DG}, \quad (5.26)$$

Combining (5.25) with (5.26) we can derive (5.24).
■

Next, we prove the main convergence result of this section.

Theorem 2. *Let $u \in W^{l,p}_{\mathcal{S}}, l \geq 2,\ p \in (\max\{1, \frac{2d}{d+2(l-1)}\}, 2]$ be the solution of (2.2a). Let $u_h \in \mathbb{B}_h(\mathcal{S}(\Omega))$ is the DGIGA solution of (3.16a) and $\Pi_h u \in \mathbb{B}_h(\mathcal{S}(\Omega))$ is the interpolant of Lemma 11. Then there are $C_i := C_i(\max_{l_0 \leq l} \|D^{l_0}\mathbf{\Phi}_i\|_{L^\infty(E)}), \|u\|_{W^{l,p}(\Omega_i)}\big)$, such that*

$$\|u - u_h\|_{DG} \leq \sum_{i=1}^{N} \Big(C_i \Big(h_i^{\delta(p,d)} + \sum_{F_{ij}\in\mathcal{F}} \alpha^{(j)} \frac{h_i}{h_j} h_i^{\delta(p,d)} \Big) \|u\|_{W^{l,p}(\Omega_i)} \Big), \tag{5.27}$$

where $\delta(p,d) = l + (\frac{d}{2} - \frac{d}{p} - 1)$.

Proof. *Since $(u_h - \Pi_h u) \in \mathbb{B}_h(\mathcal{S}(\Omega))$ by the discrete coercivity (4.21) we have*

$$\|u_h - \Pi_h u\|^2_{DG} \leq a_h(u_h - \Pi_h u, u_h - \Pi_h u). \tag{5.28}$$

By orthogonality we have

$$\begin{aligned}
&\|u_h - \Pi_h u\|^2_{DG} \leq a_h(u_h - \Pi_h u, u_h - \Pi_h u) = \\
&\quad a_h\big((u_h - u) + (u - \Pi_h u), u_h - \Pi_h u\big) = a_h\big(u - \Pi_h u, u_h - \Pi_h u\big) \\
&\quad \leq C\Big(\|u - \Pi_h u\|_{DG} + \Big(\sum_{F_{ij}\in\mathcal{F}} h_i^{1+\gamma_{p,d}} \alpha^{(i)} \|\nabla u^{(i)} - \Pi_h u^{(i)}\|^p_{L^p(F_{ij})} \\
&\qquad\qquad + h_j^{1+\gamma_{p,d}} \alpha^{(j)} \|\nabla u^{(i)} - \Pi_h u^{(j)}\|^p_{L^p(F_{ij})} \Big)^{\frac{1}{p}} \Big) \|u_h - \Pi_h u\|_{DG},
\end{aligned}$$

where immediately we get

$$\begin{aligned}
\|u_h - \Pi_h u\|_{DG} \leq \|u - \Pi_h u\|_{DG} + \Big(\sum_{F_{ij}\in\mathcal{F}} h_i^{1+\gamma_{p,d}} \alpha^{(i)} \|\nabla u^{(i)} - \Pi_h u^{(i)}\|^p_{L^p(F_{ij})} \\
+ h_j^{1+\gamma_{p,d}} \alpha^{(j)} \|\nabla u^{(i)} - \Pi_h u^{(j)}\|^p_{L^p(F_{ij})} \Big)^{\frac{1}{p}}.
\end{aligned} \tag{5.29}$$

Now, using triangle inequality, the approximation estimates (5.3) and the bound (5.16) in (5.29), we obtain

$$\begin{aligned}
&\|u_h - u\|_{DG} \leq \|u_h - \Pi_h u\|_{DG} + \|u - \Pi_h u\|_{DG} \leq \\
&\quad \sum_{i=1}^{N} C_i h_i^{\delta(p,d)} \|u\|_{W^{l,p}(\Omega_i)} + \sum_{F_{ij}\in\mathcal{F}} C_i \frac{\alpha^{(j)} h_i}{h_j} h_i^{\delta(p,d)} \|u\|_{W^{l,p}(\Omega_i)},
\end{aligned} \tag{5.30}$$

which is the required error estimate (5.27).

■

6 Numerical examples

In this section, we present a series of numerical examples to validate numerically the theoretical results, which were previously shown. We restrict ourselves for a model problem in $\Omega = (\frac{-1}{2}, \frac{1}{2})^{d=3}$, with $\Gamma_D = \partial\Omega$. The domain Ω is subdivided in four equal sub-domains $\Omega_i, i = 1, ..., 4$, where for simplicity every Ω_i is initially partitioned into a mesh $T^{(i)}_{h_i,\Omega_i}$ with $h := h_i = h_j, i \neq j, i, j = 1, ..., 4$. Successive uniform refinements are performed on every $T^{(i)}_{h_i,\Omega_i}$ in order to compute numerically the convergence rates. We set the diffusion coefficient equal to one.

All the numerical tests have been performed using G+SMO[1], which is a generic object oriented C++ library for IGA computations. In the first test, the data u_D and f in (2.1) are determined so that the exact solution is given by $u(x) = \sin(2.5\pi x)\sin(2.5\pi y)\sin(2.5\pi z)$ (highly smooth test case). The first two columns of Table 1 display the convergence rates. As it was expected, the convergence rates are optimal. In the second case, the exact solution is $u(x) = |x|^\lambda$. The parameter λ is chosen such that $u \in W^{l,p=1.4}(\Omega)$, [37]. In Table 1 in the last columns, we display the convergence rates for degree $k = 2,\ k = 3$ and $l = 2,\ l = 3$. We observe that, for each of the two different tests, the error in the $\|.\|_{DG}$ norm behaves according to the main error estimate given by (5.27).

	highly smooth		$k=2$		$k=3$	
$\frac{h}{2^s}$	$k=2$	$k=3$	$l=2$	$l=3$	$l=2$	$l=3$
-	Convergence rates					
$s=0$	-	-	-	-	-	-
$s=1$	0.15	2.91	0.62	0.76	0.24	1.64
$s=2$	2.34	2.42	0.29	1.10	0.28	0.89
$s=3$	2.08	3.14	0.35	1.32	0.47	1.25
$s=4$	2.02	3.04	0.35	1.36	0.36	1.37

Table 1. The numerical convergence rates of the DGIGA method.

Remark 2. In a forthcoming paper, we will present graded mesh techniques in DGIGA methods for treating problems with low regularity solutions. We will show, how to construct graded refined mesh in the vicinity of the singular points of u, in order to get the optimal approximation order as in the case of having smooth u.

[1] G+SMO: http://www.gs.jku.at/gs-gismo.shtml

7 Conclusions

In this paper, we presented theoretical error estimates of the DGIGA method applied to a model elliptic problem with discontinuous coefficients. The problem was discretized according to IGA methodology using discontinuous $\mathbb{B}$-Spline spaces. Due to global discontinuity of the approximate solution on the sub-domain interfaces, DG discretizations techniques were utilized. In the first part, we assumed higher regularity for the exact solution, that is $u \in W^{l\geq 2,2}$, and we showed optimal error estimates with respect to $\|.\|_{DG}$. In the second part, we assumed low regularity for the exact solution, that is $u \in W^{l\geq 2,p\in(\frac{2d}{d+2(l-1)},2)}$, and applying the Sobolev embedding theorem we proved optimal convergence rates with respect to $\|.\|_{DG}$. The theoretical error estimates were validated by numerical tests.

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